Arrangements

Overview

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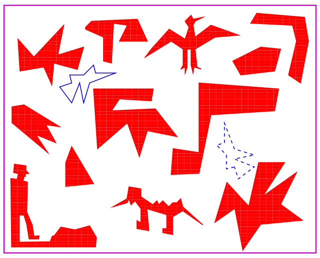
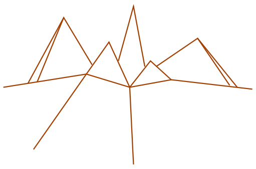
Problem 1: Motion planning



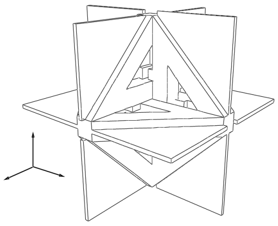
decide whether a collision-free motion for the moving object from start to goal exists, and if so plan the motion

Problem 2: Aspect graph of a terrain



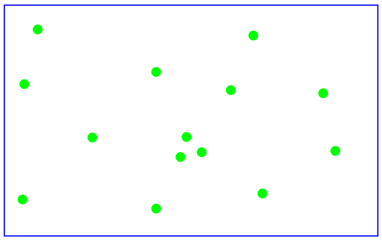
design a compact representation of all the different 2D images of a polyhedral terrain, so that the view in a given query direction can be efficiently retrieved

Problem 3: Is the 3D object interlocked?



decide whether an assembled object is interlocked, namely cannot be taken apart with two hands

Problem 4: Minimum area triangle



find the three of the given points that define the minimum area triangle

Q:

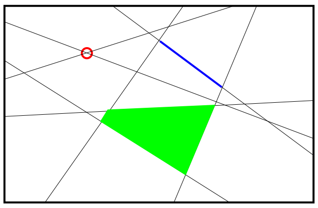
what is the connection between Problems 1,2,3 and 4?

A:

the best solution known to each of them was obtained with arrangements

What are arrangements?

Example: an arrangement of lines



vertex

edge

face

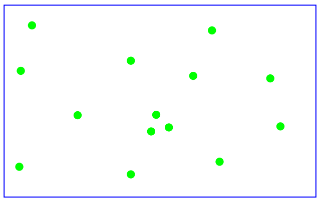
What are arrangements, cont'd

- an arrangement of a set S of geometric objects is the subdivision of space where the objects reside induced by S
- possibly non-linear objects (parabolas), bounded objects (segments, circles), higher dimensions (planes, simplices)
- numerous applications in robotics, molecular biology, vision, graphics, CAD/CAM, statistics, GIS
- have been studied for decades, originally mostly combinatorics
  Matoušek (2002) cites Steiner,1826
  nowadays mainly studied in combinatorial and computational geometry

Solving it with arrangements

- transforming to arrangements
- combinatorial analysis
- design of data structures / algorithms
- implementation

A tour of “solving it with arrangements”: The minimum area triangle



find the three of the given set of n points $P = \{p_1, p_2, \dots, p_n\}$ that define the minimum area triangle

a naive algorithm requires $O(n^3)$ time

The transformation: Duality

primal plane	dual plane
the point $p := (a, b)$	the line $p^* := (y = ax - b)$
the line $l := (y = cx + d)$	the point $l^* := (c, -d)$

this duality transform does not handle vertical lines

Properties of this duality transform

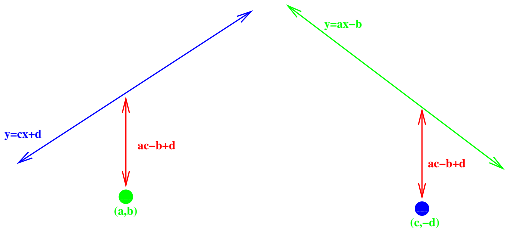
preserves incidence

$\Uparrow$

preserves above/below relation

$\Uparrow$

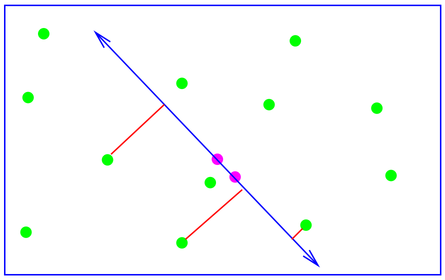
preserves the vertical distance between a line and a point



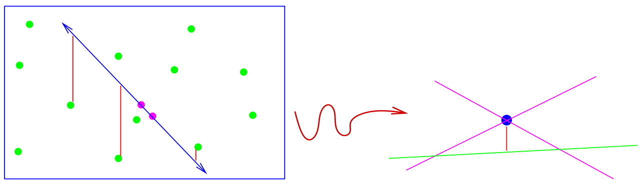
Minimum area triangle for a fixed pair

fix a pair of input points p_i, p_j

which point p_k of P defines the smallest area triangle with p_i, p_j?



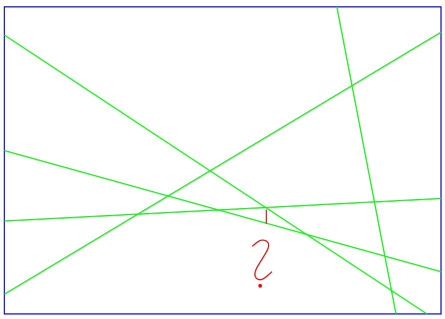
. . . in the dual



in an arrangement of n lines $P^* = \{p_1^*, p_2^*, \ldots, p_n^*\}$, for each vertex find the vertically closest line

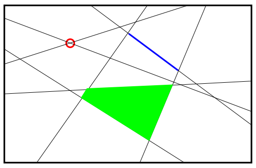
Interim summary

our problem: given an arrangement of n lines, for each vertex find the line vertically closest to it



we have reduced the number of candidate triples to be $O(n^2)$

Arrangements of lines: Combinatorics



the complexity of an arrangement is the overall number of cells of all dimensions comprising the arrangement

for planar arrangements we count: vertices, edges, and faces

the general position assumption: two lines meet in a single point, three lines have no point in common

In an arrangements of n lines

number of vertices: $n(n-1)/2$

number of edges: n^2

number of faces:
using Euler's formula $\#V - \#E + \#F = 2$

we get $n^2/2 + n/2 + 1$

Basic theorem of arrangement complexity

the maximum combinatorial complexity of an arrangement of n well-behaved curves in the plane is $O(n^2)$; there are such arrangements whose complexity is $\Omega(n^2)$

more generally

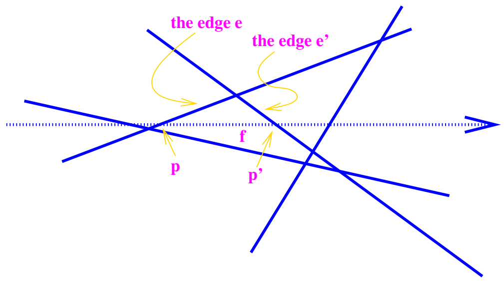
the maximum combinatorial complexity of an arrangement of n well-behaved (hyper)surfaces in $\mathbb{R}^d$ is $O(n^d)$; there are such arrangements whose complexity is $\Omega(n^d)$

Constructing the arrangement

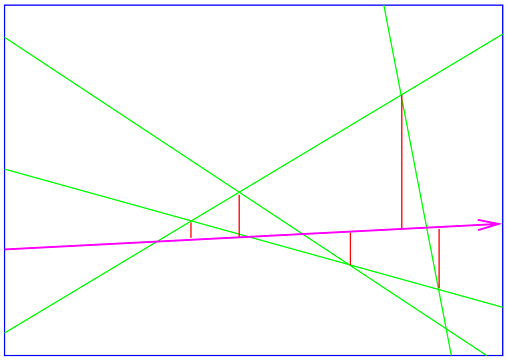
- representation (data structure)
  - DCEL - the doubly connected edge list
- algorithm
  - incremental
- computational model
  - the real RAM
- recall the general position assumption

Incremental construction

- computing a bounding box
- inserting the ith line



Searching for the minimum area triangle



line by line, going over all the relevant vertices

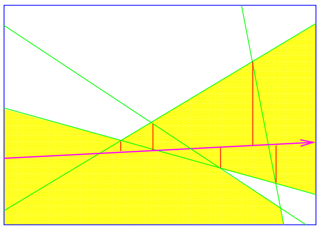
How much time does it take?

the steps:

- computing the bounding box
naively $O(n^2)$; can be done in $O(n \log n)$, exercise
- finding where to insert line i
simple, $O(i)$
- inserting line i
O(zone complexity)
- searching for the minimum area triangle for one line
O(zone complexity)

The zone of a curve

the zone of a curve γ in an arrangement $\mathcal{A}$ is the collection of faces of $\mathcal{A}$ intersected by γ



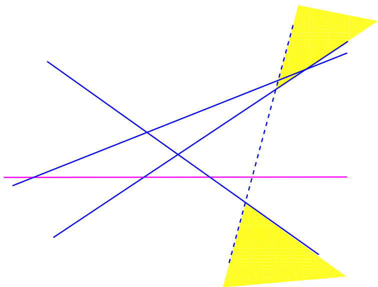
the complexity of the zone is the overall complexity of cells of various dimensions in the closure of the zone

we need: the complexity of the zone of a line in an arrangement of i lines

Zone theorem

theorem: the complexity of the zone of a line in an arrangement of i lines is $O(i)$

proof:



Overall running time of the algorithm

- computing the bounding box
naively $O(n^2)$; can be done in $O(n \log n)$
- finding where to insert line i
simple, $O(i)$
- inserting line i
$O(\text{zone complexity}) = O(i)$
- searching for the minimum area triangle for vertices on one line
$O(\text{zone complexity}) = O(i)$

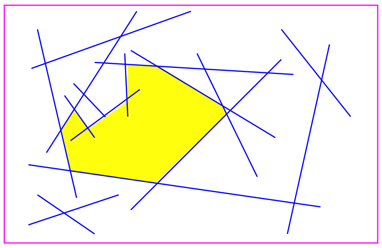
overall $O(n^2)$ time

Basics of arrangements, summary

- the combinatorial complexity of an arrangement
- incremental construction
- the zone of an(other) object in an arrangement
- the general position assumption
- the real RAM model

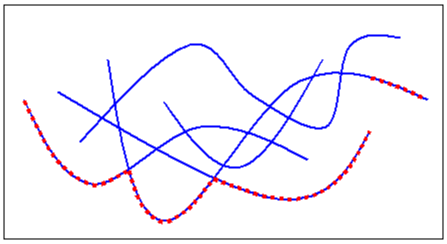
More substructures

a single face



another important substructure of arrangements: the lower (or upper) envelope

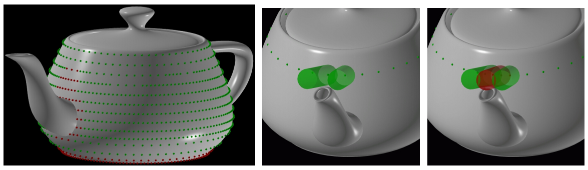
(Lower) Envelopes

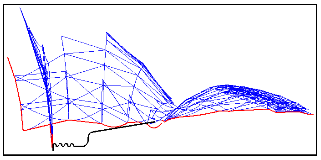


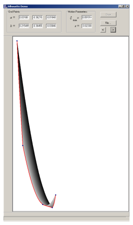
Example (recent) application of envelopes

Continuous path verification in multi-axis NC machining

[Wein-H-Ilushin-Elber 04]







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Problem 4 (min area triangle), notes

- the solution to the minimum-area-triangle problem [Chazelle-Guibas-Lee 84]
- the solution for any fixed dimension (minimum volume simplex) appears in Edelsbrunner's book (1987)
- the efficiency of the solution in any dimension relies on a hyperplane zone theorem [Edelsbrunner-Seidel-Sharir, 1993]
- no better solution is known to the problem; related to the so-called 3-sum hard problems
- the solution of Problems 1–3 will (hopefully) be discussed later on

THE END