# More on Exercise 2.3

We start with a reminder.

**Exercise 2.3: Largest common point sets under $\varepsilon$-congruence** Given two finite sets of points $A$ and $B$ in the plane, we wish to find equally sized subsets $A' \subseteq A$ and $B' \subseteq B$ of maximal cardinality such that points in $A'$ match points in $B'$ under translation, up to distance at most some given $\varepsilon$. Namely, each point $a$ in $A'$ has a unique point $b$ in the translated $B'$ such that the Euclidean distance between $a$ and $b$ is at most $\varepsilon$ and $A'$ is the largest cardinality such subset.

More technical details appear in the original assignment document, and in the course's site. Here we outline a possible direction for solution.

As we typically approach a variety of problem in computational geometry, the first step is to rephrase the problem in a "convenient" space where we can get helpful insights. In this case this is the space of all possible translations in the plane, which we denote by $\mathcal{T}$, and which is two dimensional. Each point $t = (x_t, y_t) \in \mathcal{T}$ represents a translation by the vector $t$.

Given a pair of points $a_i \in A$ and $b_j \in B$ we denote by $D_{ij}$ all the translation vectors which when applied to $b_j$ will make it $\varepsilon$-congruent to $a_i$, that is

$$D_{ij} = \{t \in \mathcal{T} | \rho(a_i, b_j + t) \leq \varepsilon\} \,,$$

where $\rho$ is the Euclidean distance between a pair of points. It is easily verified that $D_{ij}$ is a disk in the translation space.

Let $m$ and $n$ denote the number of points in $A$ and $B$ respectively. Let $\mathcal{D}$ denote the collection of the $mn$ disks $D_{ij}$ induced by all pair of points one from $A$ and one from $B$. Let $C_{ij}$ be the boundary circle of $D_{ij}$, and let $\mathcal{C}$ be the collection of the $mn$ circles $C_{ij}$. Our main object of study is the arrangement $\mathcal{A}(\mathcal{C})$ of circles in the translation space. (We will refer both to the disks and to the circles, as needed.)

Consider now the following question, which is a restricted version of the Exercise, where we fix the translation. Given a translation vector $t_0$, find equally sized subsets $A' \subseteq A$ and $B' \subseteq B$ of maximal cardinality such that points in $A'$ match points in $B'$ under translation *by the vector $t_0$*, up to distance at most $\varepsilon$.

The solution can be obtained as follows. We construct a bipartite graph: The nodes of the graph correspond to points of $A$ and $B$. We connect two nodes corresponding to the points $a_i$ and $b_j$ iff $b_j + t_0 - a_i \in D_{ij}$. That is, we go over all the disks in $\mathcal{D}$, and for each disk that contains $t_0$ (either inside or on its boundary) we add the respective edge to the graph. It remains to find a maximum matching in this graph, which will yield the desired solution.

We have to repeat this process for every translation vector $t \in \mathcal{T}$. A nice feature of the arrangement $\mathcal{A}(\mathcal{C})$ defined above is: *Given any fixed face of the arrangement, all the points inside it induce exactly the same bipartite graph.* This means that the arrangement $\mathcal{A}(\mathcal{C})$ gives us an effective discretization of the problem without giving up any information. Now it remains to go over all the faces of the arrangement, and for each face sample one point, construct its relevant bipartite graph and proceed as in the single vector case described above.

This process can be sped up by observing that between two neighboring faces the graph changes only a little. Also, a closer inspection of the problem reveals that it suffices to check only *convex* faces of the arrangement. (Interestingly however, there can be asymptotically as many convex faces in the arrangement as there are non-convex faces.)