Envelopes

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Problem 1: Largest scatter

$P = \{p_1(t), \ldots, p_n(t)\}$ a set of n moving points in $\mathbb{R}^2$
point coordinates: polynomial functions of t of max degree s

find the time when P is most scattered:
smallest distance is as large as possible

Problem 2: Nearest Jeep over time

n Jeeps are driving in a flat desert, each in a straight lane and fixed velocity (the lanes or velocities are not necessarily the same); find the nearest Jeep to the base station over time

Problem 3: Collision detection in NC machining

decide whether the tool at a given position intersects the workpiece or other parts of the machine

Problem 4: Visibility map

build a map describing the visible parts of a scene of triangles in space (hidden surface removal plus workable map)

Davenport-Schinzel sequences

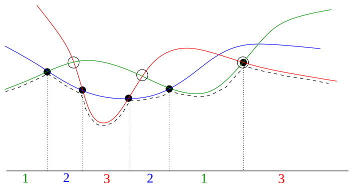
- U is a word composed of n symbols
- no symbol appears twice in a row
- No pair of distinct symbols a, b alternate in U $s+2$ times, namely no $(a \cdots b \cdots a \cdots b \cdots)$ of length $s+2$

then U is a Davenport-Schinzel sequence of order s on n symbols, denoted $DS(n,s)$-sequence

$\lambda_s(n)$ = maximum length of such a sequence

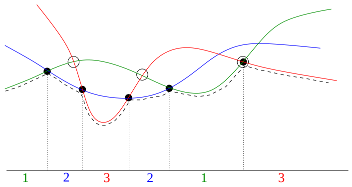
Lower envelopes

$\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ real-valued continuous functions defined over a common interval I, each pair of which intersect in at most s points



the lower envelope Ψ of the collection $\mathcal{F}$ is the pointwise minimum of these functions:
$\Psi(x) = \min f_i(x), x \in I$
similarly, the upper envelope is defined as the pointwise maximum of these functions

Minimization diagrams



the minimization diagram: the subdivision of the x-axis into maximal intervals so that on each interval the same subset of functions attains the minimum

let $I_1, \ldots, I_m$ be the maximal intervals of I such that a single function attains the minimum on this interval, ordered from left to right
let $U(f_1, \ldots, f_n) =< u_1, \ldots, u_m >$ be the indices of these functions

Lower envelopes and Davenport-Schinzel sequences

claim 1: $U(f_1, \ldots, f_n)$ is a $DS(n, s)$-sequence

claim 2: for any given $DS(n, s)$-sequence U, there is a collection $\mathcal{F}$ as above such that $U(f_1, \ldots, f_n) = U$

the maximum length of $U(f_1, \ldots, f_n)$ is $\lambda_s(n)$

Partially defined functions

$\mathcal{G} = \{g_1, g_2, \ldots, g_n\}$ partially defined real-valued continuous functions, each pair of which intersect in at most s points

$U(g_1, \ldots, g_n)$ is a $DS(n, s+2)$-sequence

for any given $DS(n, s+2)$-sequence U, there is a collection $\mathcal{G}$ as above such that $U(g_1, \ldots, g_n) = U$

the lower envelope of n non-vertical segments in the plane is a $DS(n, 3)$-sequence

Functions intersecting a common vertical line

$\mathcal{H} = \{h_1, h_2, \ldots, h_n\}$ partially defined real-valued continuous functions, each pair of which intersect in at most s points, and there exists a vertical line that intersects all their graphs. then the maximum length of $U(h_1, \ldots, h_n)$ is $2\lambda_{s+1}(n)$

exercise

Bounds on $\lambda_s(n)$

easy:

- $\lambda_1(n) = n$
- $\lambda_2(n) = 2n - 1$
- $\lambda_3(n) \leq 2n(\ln n + O(1))$

$\lambda_s(n) = O(n \log^* n)$, for a fixed s [Szemerédi '74]
(improved bounds -> the next slide)

Bounds on $\lambda_s(n)$

- $\lambda_3(n) = \Theta(n\alpha(n))$
- $\lambda_4(n) = \Theta(n \cdot 2^{\alpha(n)})$
- $\lambda_{2s}(n) = n \cdot 2^{\Theta(\alpha^{s-1}(n))}$

$\alpha(\cdot)$: the functional inverse of Ackermann's function, extremely slowly growing; $\alpha(n) \leq 4$, for n up to a tower with 65536 2's
[Hart-Sharir '86], [Agarwal-Sharir-Shor '89]

if the function graphs are n line segments ($s = 3$ after making them totally defined), maximum complexity of lower envelope is $\Theta(n\alpha(n))$!!! [Wiernik-Sharir], [Shor]

Algorithm 1: Divide-and-conquer

assumptions

- real RAM model
- curves are well-behaved
- each pair intersecting in at most s points

running time

- for totally defined functions: $O(\lambda_s(n) \log n)$
- for partially defined functions: $O(\lambda_{s+2}(n) \log n)$

Algorithm 2: slight improvement

for partially defined functions an improved algorithm has running time $O(\lambda_{s+1}(n) \log n)$ [Hershberger '89]

in particular for n line segments, th envelope can be computed in $O(n \log n)$ time

Solving Problem 1

reminder

Problem 1, largest scatter: $P = \{p_1(t), \ldots, p_n(t)\}$ a set of n moving points in $\mathbb{R}^2$,

point coordinates—polynomial functions of t of max degree s

find the time when P is most scattered, namely smallest distance is as large as possible

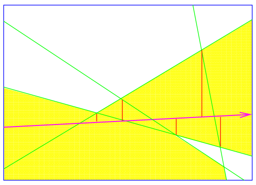
define $f_{ij}(t) = \|p_i(t) - p_j(t)\|^2$, $\quad 1 \le i < j \le n$

polynomials of max degree $2s$

construct the lower envelope of the graphs of the $f_{ij}(t)$'s and find the maximum

Alternative proof of the zone theorem

the zone of a curve γ in an arrangement $\mathcal{A}$ is the collection of faces of $\mathcal{A}$ intersected by γ



the complexity of the zone is the overall complexity of cells of various dimensions in the closure of the zone

an alternative proof of the linear bound on the zone complexity of a line in an arrangement of lines, using DS-sequences

Envelopes in higher dimensions

the complexity of the lower envelope of n d-variate functions, whose graphs are semi-algebraic sets of constant description complexity, is $O(n^{d+\varepsilon})$, for any $\varepsilon > 0$ [H-Sharir, 1994], [Sharir, 1994]

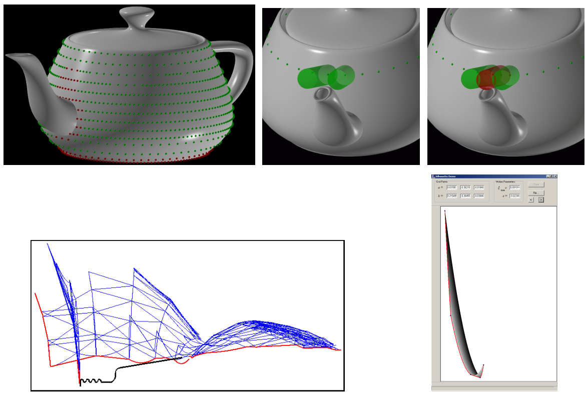
near-quadratic time divide-and-conquer algorithm for bivariate functions (lower envelopes in 3D), based on near-quadratic overlay bound

Example (recent) application of envelopes

Continuous path verification in multi-axis NC machining

(a difficult variant of Problem 3)

[Wein-H-Ilushin-Elber 04]



A variety of applications

- general Voronoi diagrams
- transversals of a set of objects
- visibility over polyhedral terrain (e.g., Problem 2 of the arrangements overview)
- dynamic geometry (e.g., convex hull of moving points)
- many more . . .

Reference

Davenport-Schinzel Sequences and Their Geometric Applications

Sharir-Agarwal, Cambridge, 1995