APPLIED aspects of COMPUTATIONAL GEOMETRY

Arrangements, 2D

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Overview

- Part I, A first tour of “solving it with arrangements”:
  The minimum area triangle problem
  - duality, dcel, incremental construction, zone
- Part II, Generalizations
  - transformations
  - different types of arrangements
  - alternative representation
  - construction by sweeping
  - other substructures: complexity and algorithms

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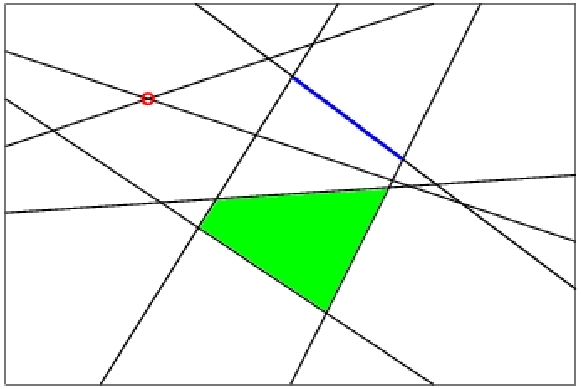
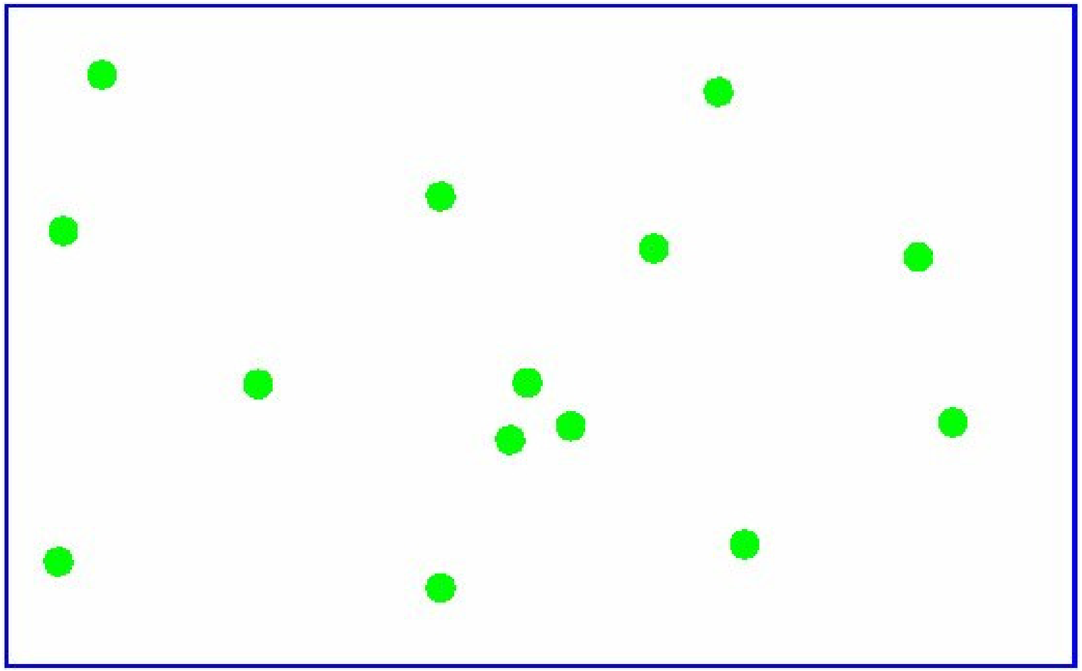
Background

- Lines, segments, and rays
- A tale of two paradigms: sweep vs. incremental
- Arrangements of lines:
  - the shape and complexity of a face
  - the complexity of the entire arrangements is $\theta(n^2)$



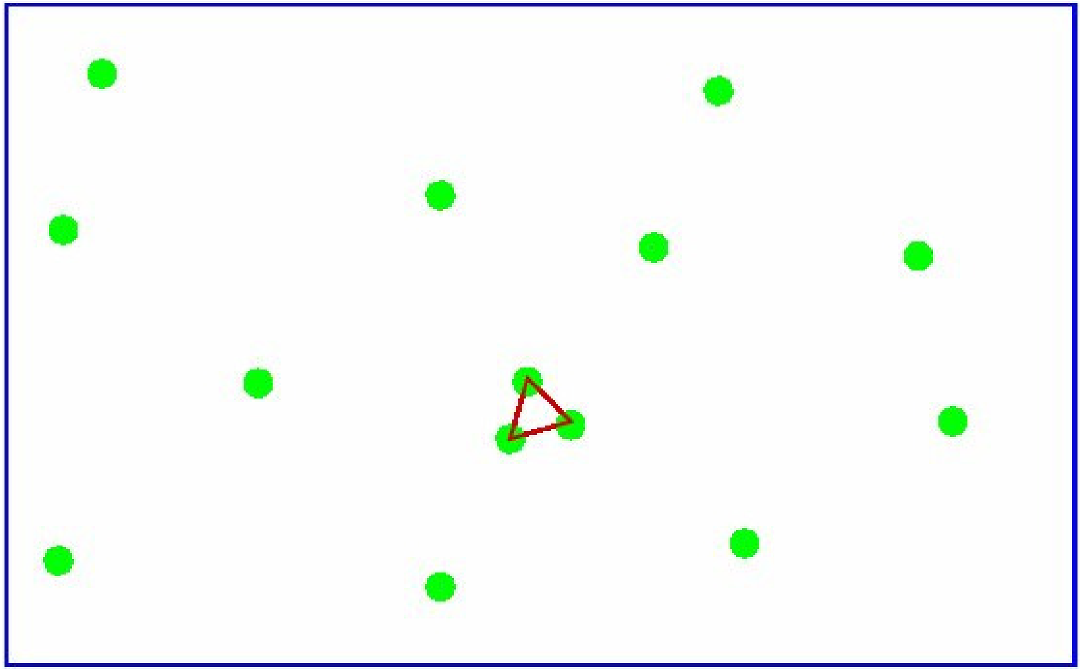
Reminder: Minimum area triangle

- find the three of the given set of n points $P = \{p_1, p_2, . . . , p_n\}$ that define the minimum area triangle



Reminder: Minimum area triangle

- find the three of the given set of n points $P = \{p_1, p_2, \ldots, p_n\}$ that define the minimum area triangle
- a naïve algorithm requires $O(n^3)$ time



The transformation: Duality

Primal plane

- the point $p := (a, b)$
- the line $l := (y = cx + d)$

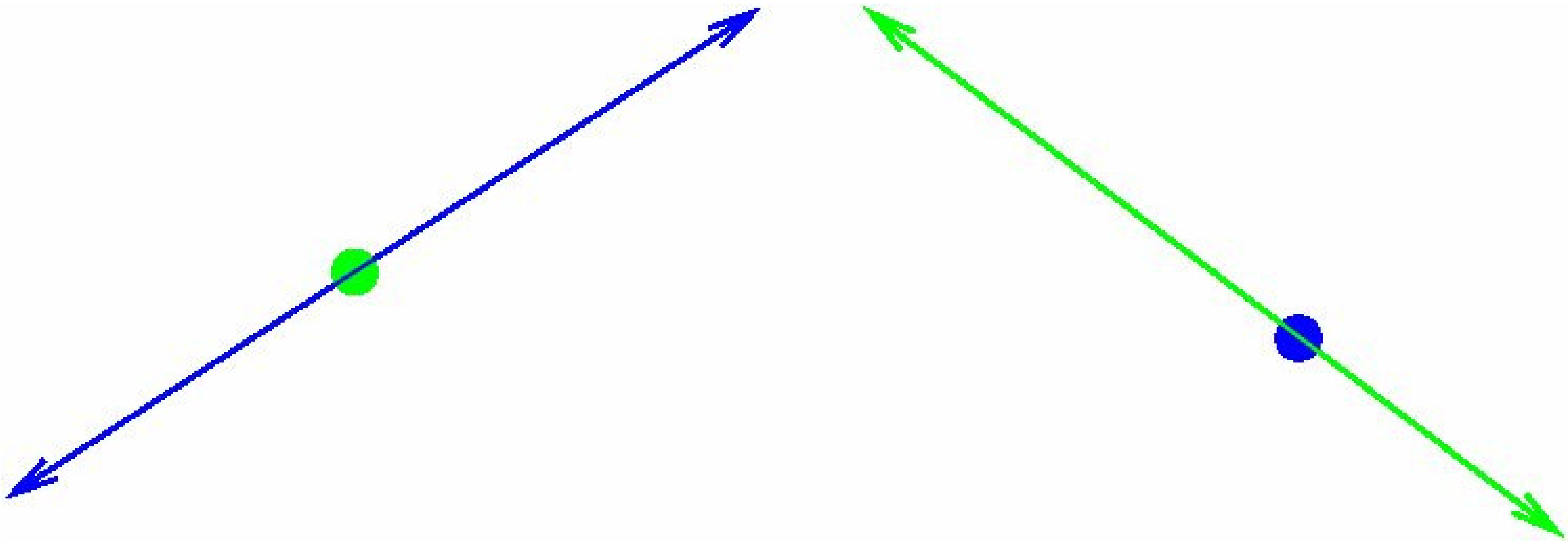
Dual Plane

- the line $p^* := (y = ax - b)$
- the point $l^* := (c, -d)$

this duality transform does not handle vertical lines

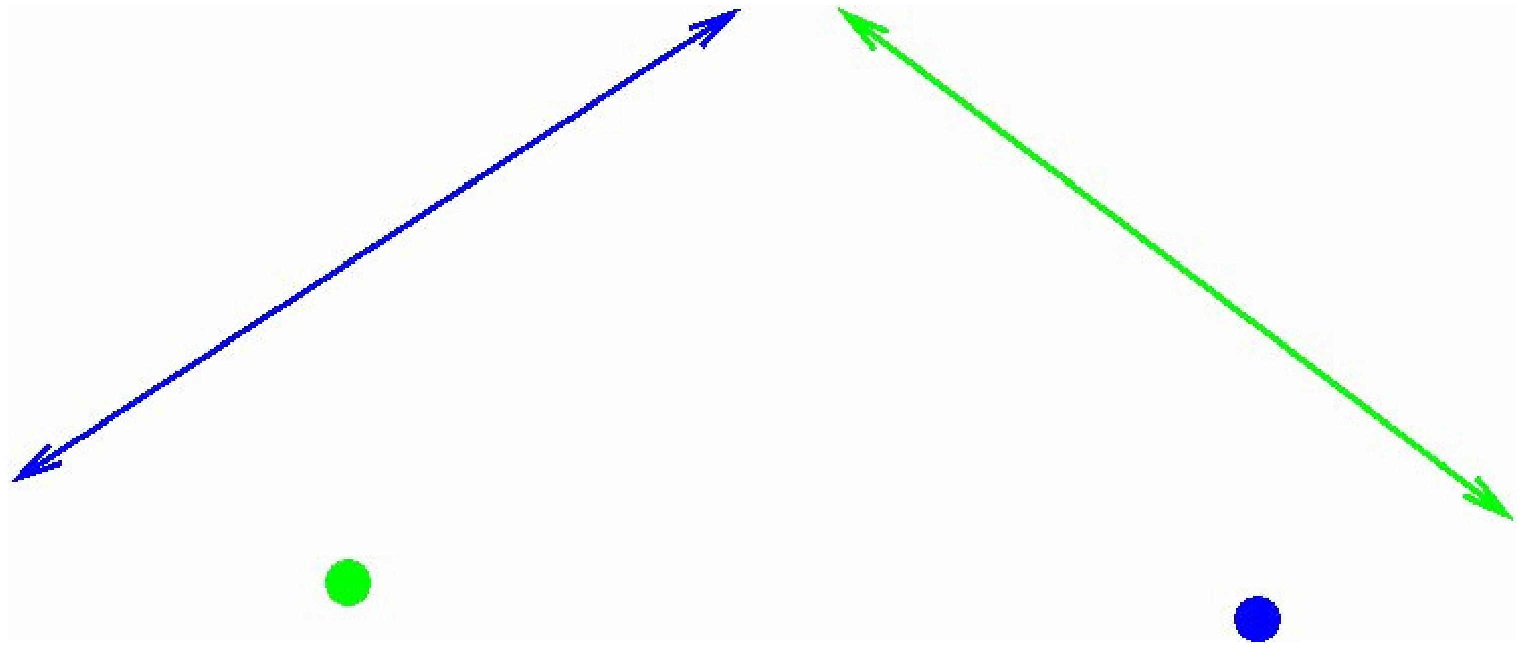
Properties of this duality transform

- preserves incidence



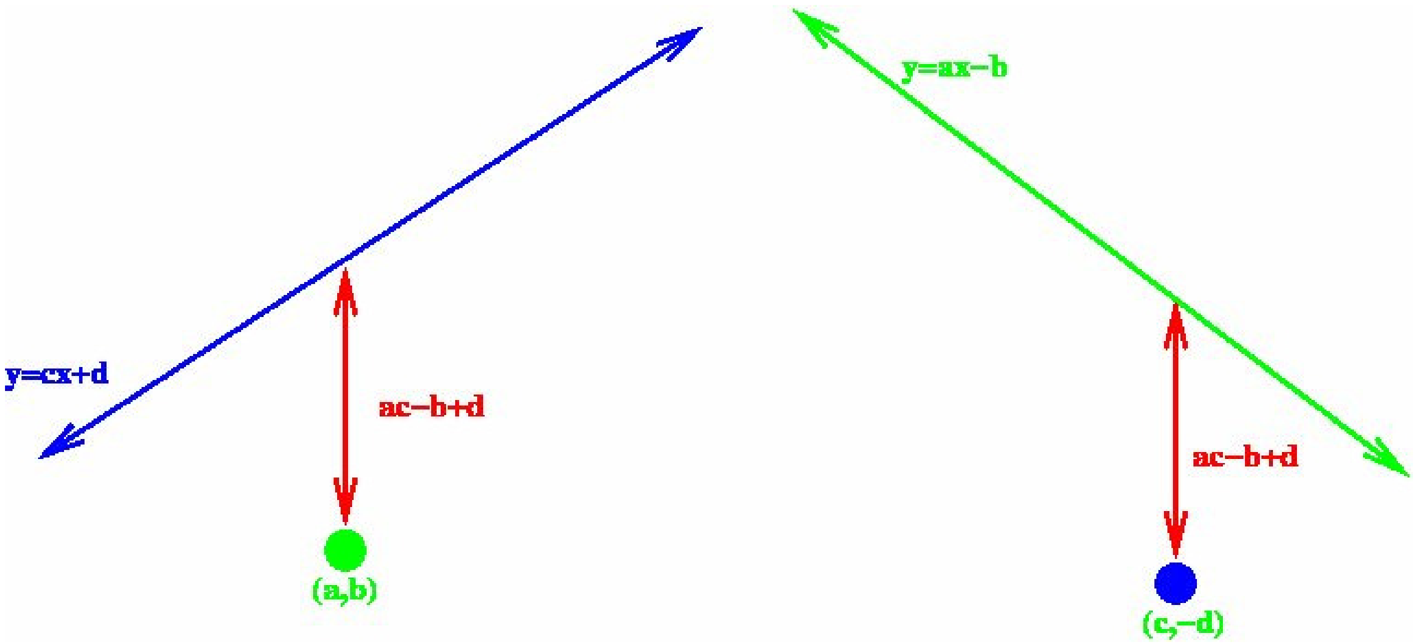
Properties of this duality transform

- preserves incidence
- preserves above/below relation



Properties of this duality transform

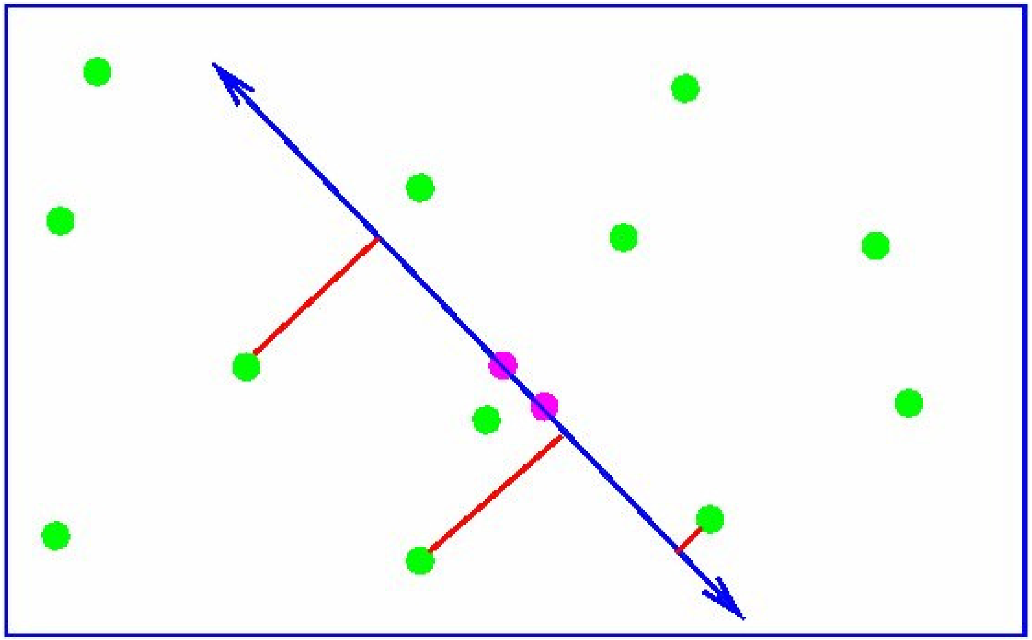
- preserves incidence
- preserves above/below relation
- preserves the vertical distance between a line and a point



Minimum area triangle for a fixed pair

- fix a pair of input points p_i ,p_j
- which point p_k of P defines the smallest area triangle

with p_i ,p_j?



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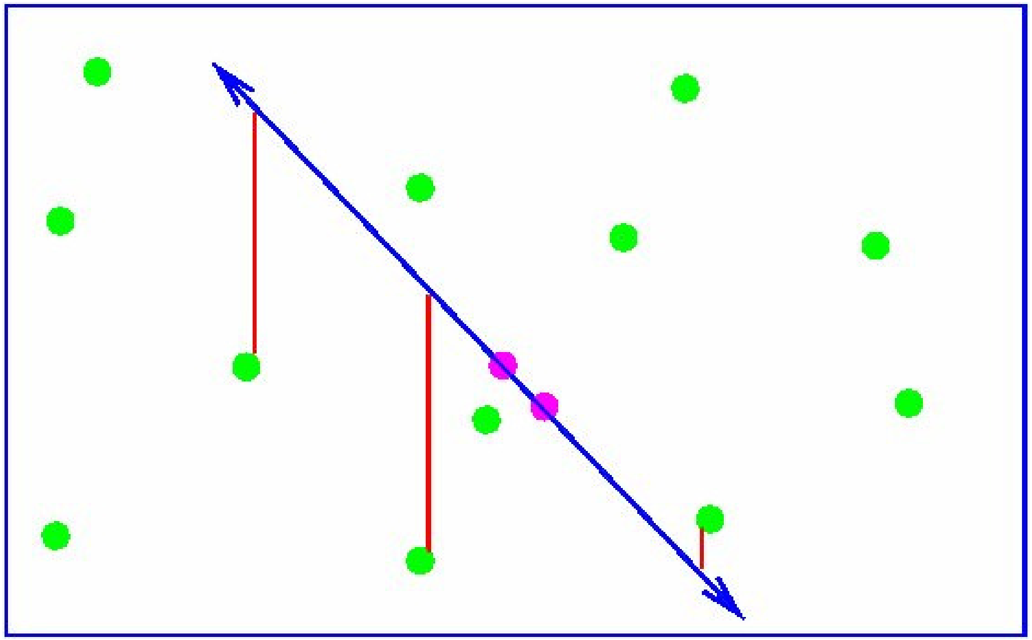
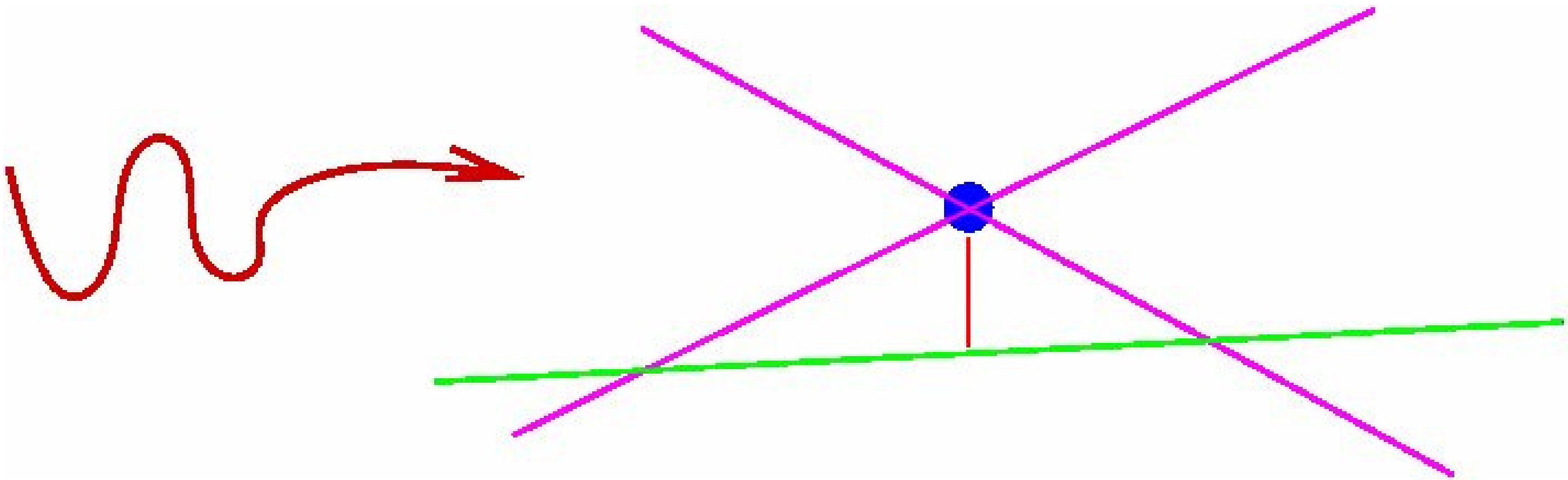


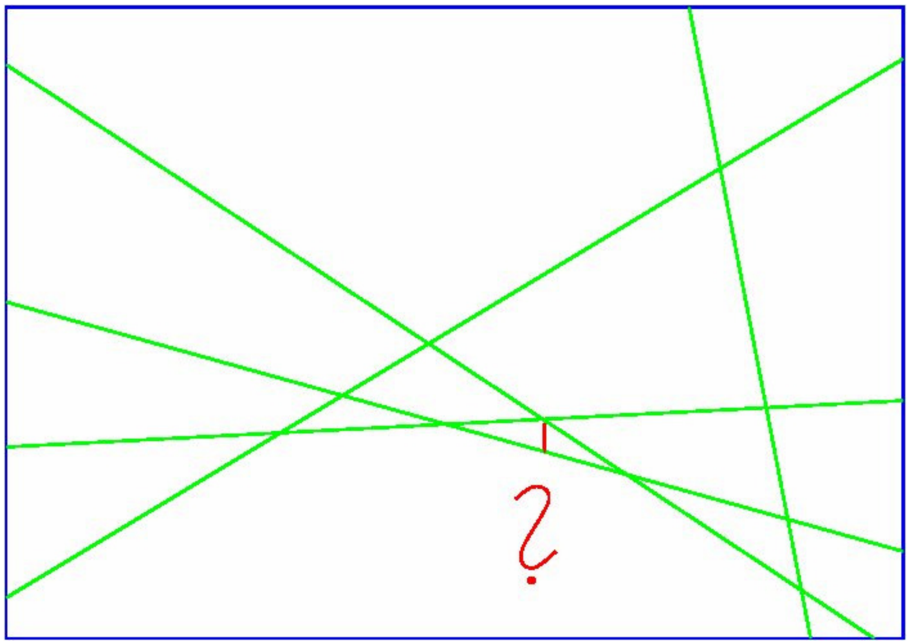
In the dual…

- in an arrangement of n lines $P^* = \{p^*_1, p^*_2, . . . , p^*_n\}$, for each vertex find the vertically closest line



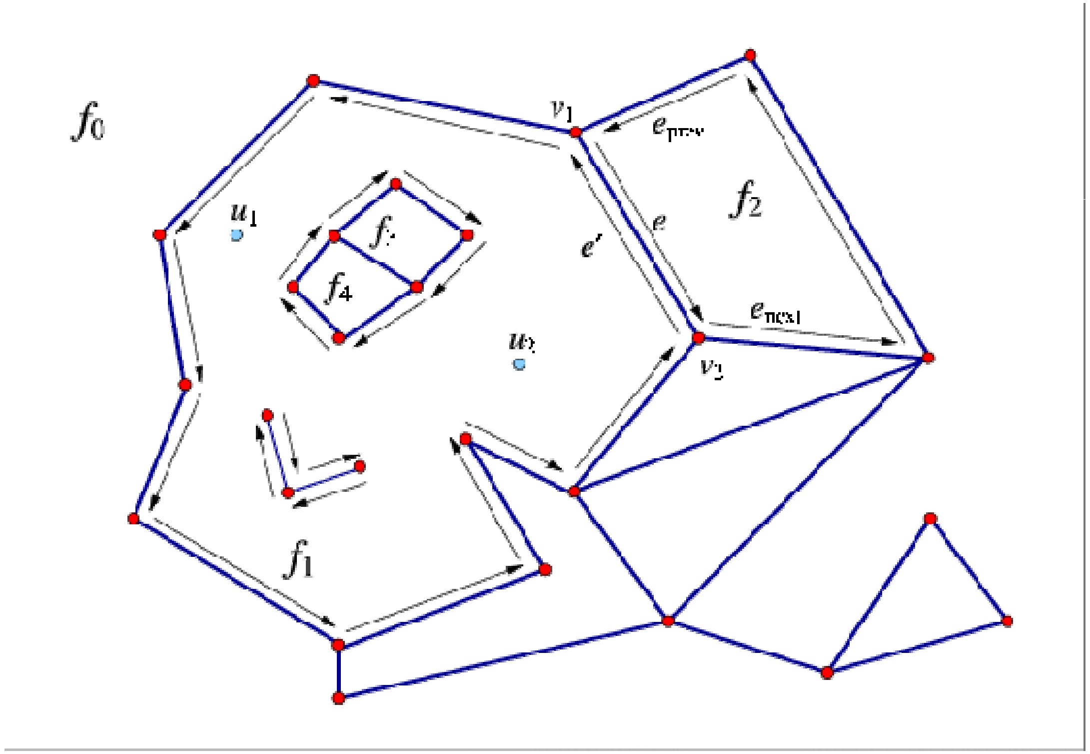
Interim summary

- our problem: given an arrangement of n lines find the vertex and the line that induce the smallest vertical distance



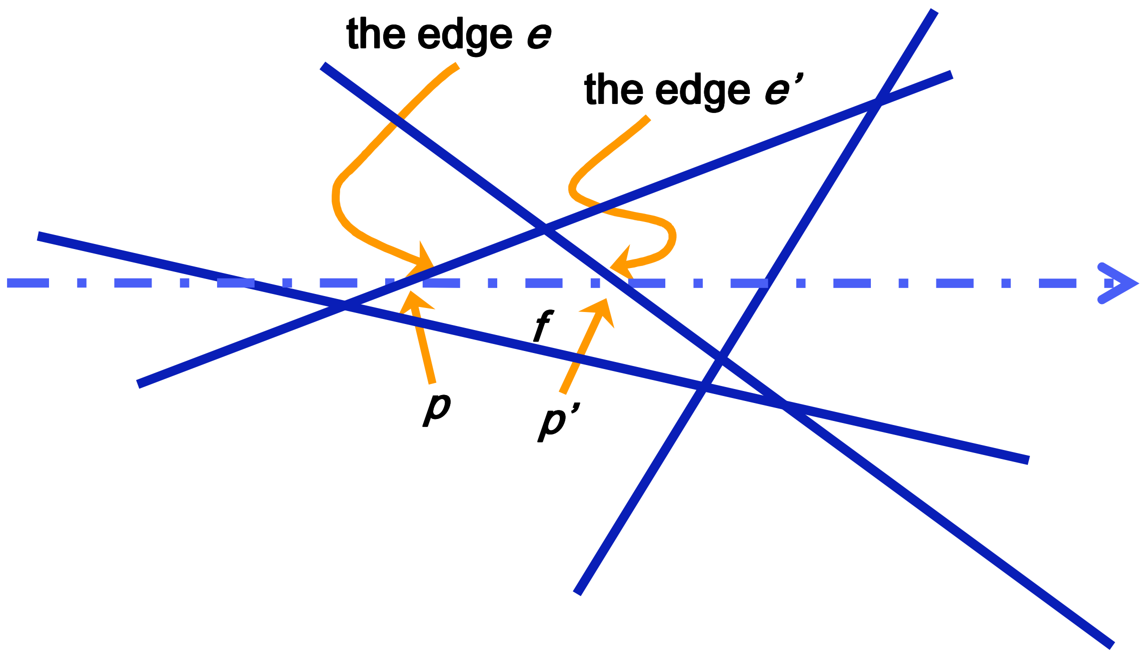
Representing the arrangement I: Dcel

- vertices, half-edges, and faces
- halfedges:
  - twin
  - previous
  - next
- CCBs: inner and outer



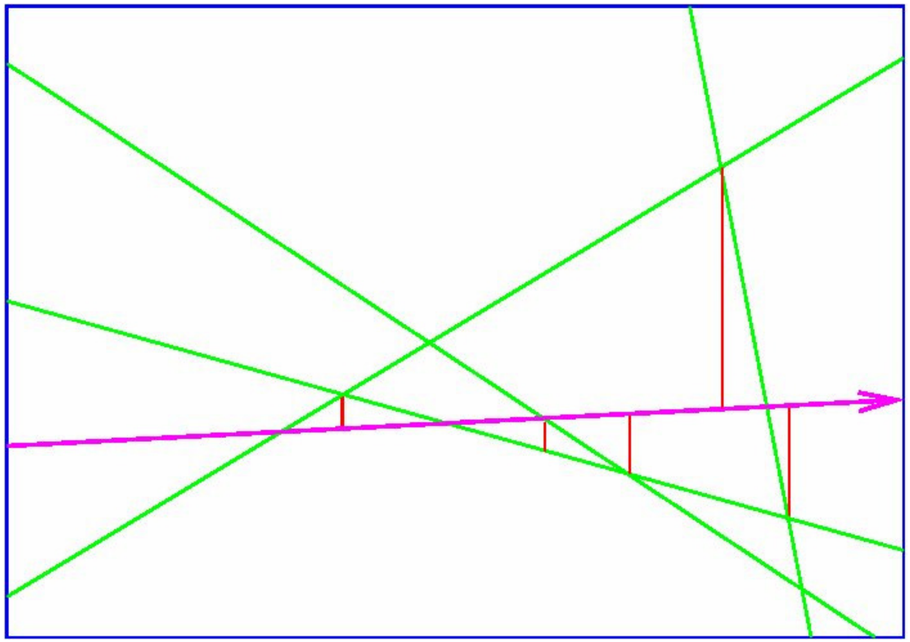
Incremental construction

- recall the general position assumption
- computing a bounding box
- inserting the *i*-th line



Searching for the minimum area triangle

- line by line, going over all the relevant vertices

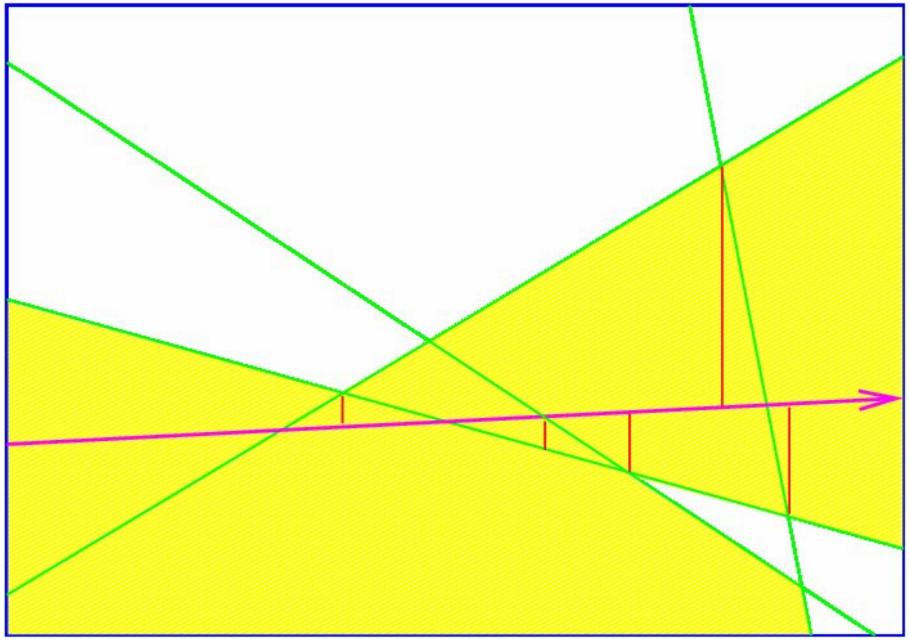


How much time does it take?

- computing the bounding box
  - naively $O(n^2)$; can be done in $O(n \log n)$
- finding where to insert line *i*
  - simple, $O(i)$
- inserting line *i*
  - *O(zone complexity)*
- searching for the minimum area triangle for one line
  - *O(zone complexity)*

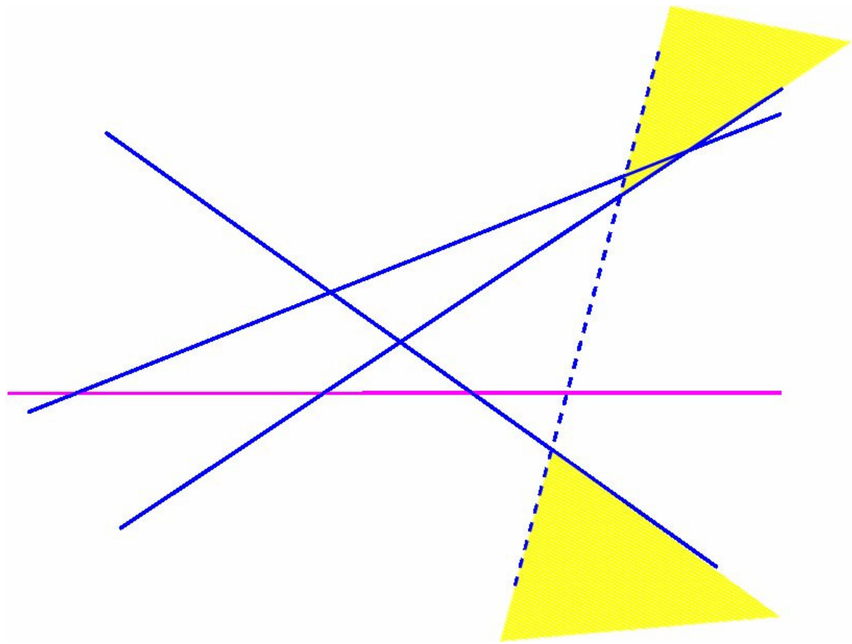
The zone of a curve

- the zone of a curve ɣ in an arrangement *A* is the collection of faces of *A* intersected by ɣ
- the complexity of the zone is the overall complexity of cells of various dimensions in the closure of the zone
- we need: the complexity of the zone of a line in an arrangement of *i* lines



Zone theorem

- theorem: the complexity of the zone of a line in an arrangement of *i* lines is *O(i)*
- proof:



How much time does it take?

- computing the bounding box
  - naively $O(n^2)$; can be done in $O(n \log n)$, exercise
- finding where to insert line i
  - simple, $O(i)$
- inserting line i
  - $O(\text{zone complexity}) = O(i)$
- searching for the minimum area triangle for one line
  - $O(\text{zone complexity}) = O(i)$
- *Overall $O(n^2)$ time*

Summary, a tour of “solving it with arrangements”: the minimum area triangle

- transforming to arrangements, duality
- combinatorial analysis zone theorem
- design of data structures: Dcel / algorithms: incremental
- implementation

Minimum area triangle, notes

- the solution to the minimum-area-triangle problem [Chazelle-Guibas-Lee 84]
- the solution for any fixed dimension (minimum volume simplex) appears in Edelsbrunner's book (1987)
- the efficiency of the solution in any dimension relies on a hyperplane zone theorem [Edelsbrunner-Seidel-Sharir 93]
- no better solution is known to the problem; related to the so-called 3-sum hard problems
- see also Ch 8 of the book *Computational Geometry* by de Berg et al, for the incremental construction of arrgs of lines

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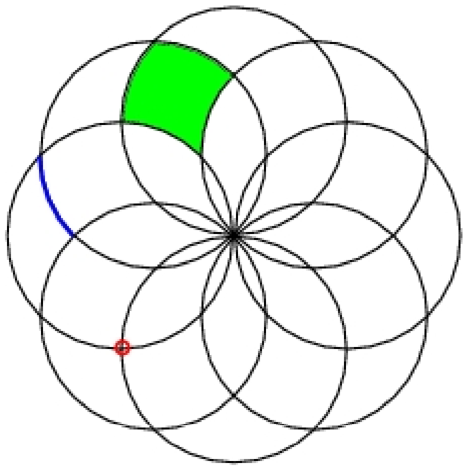
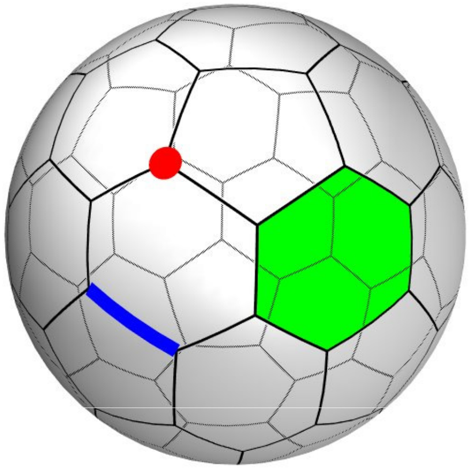
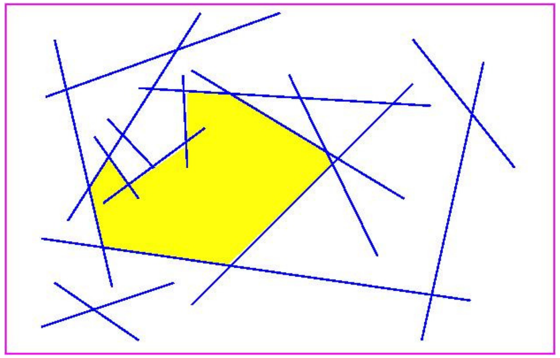
Transformation to arrangements

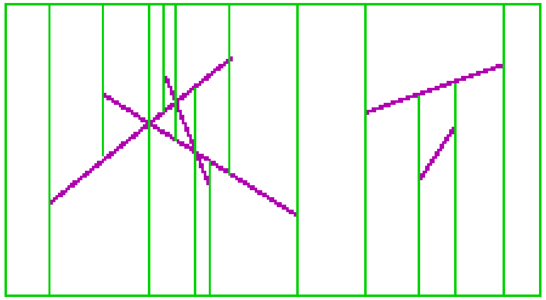
- various dualities, Plücker coordinates, the locus method (configuration space, Minkowski sums), and numerous other
- we will see a few more later in the course

Variety of 2D arrangements

- major complications
  - faces can have convoluted shapes
  - the algebra becomes more involved

Vertical decomposition



- the complexity in the plane O(arrg-complexity)
  - n interior-pairwise-disjoint segments in a bounding box

    ⟶ at most 3n+1 trapezoids
- extends to higher dimensions and “well-behaved” surfaces
- partial decomposition

Construction by sweeping

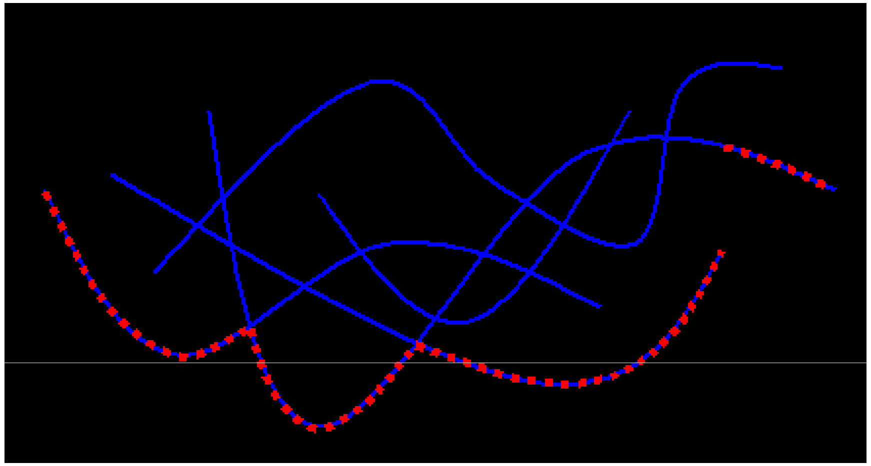
- sweeping a vertical line
- status line: intersecting curves in order
- events: endpoints and intersection points
- event queue
- complexity for arrg of n “well-behaved” curves: O(arrg-complexity x log n)
- a possible by-product within the same running time: the vertical decomposition

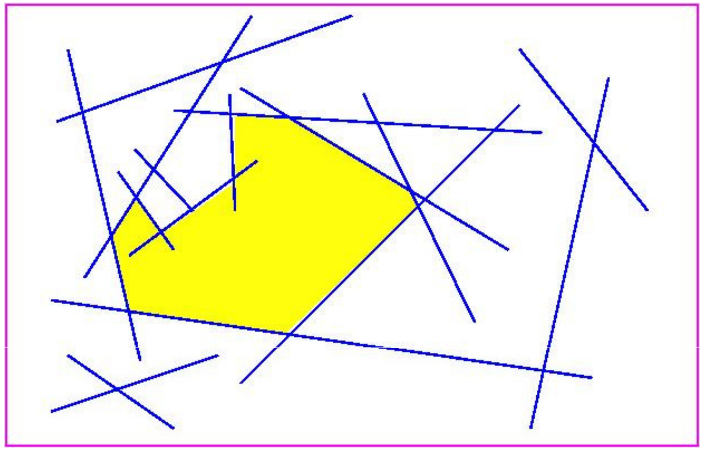
Sweep vs. incremental construction

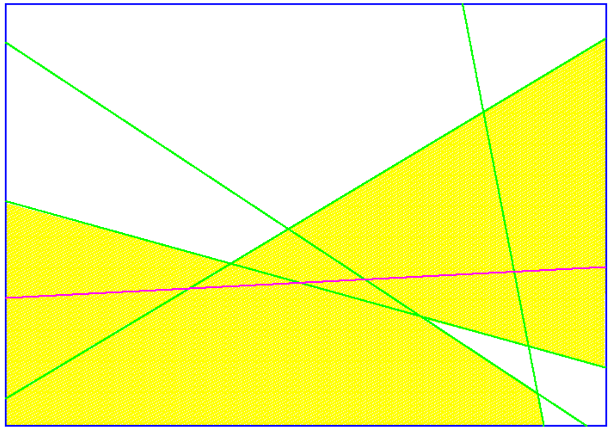
- efficiency (in theory) for line arrangement
- what about other types of arrangements?
  - the sweep has the same complexity for n “well-behaved” curves (constant # of pairwise intersections): O(arrg-complexity x log n)
  - for incremental construction:

    zone theorem for curves?

Substructures: envelope, single face, zone







Davenport-Schinzel sequences

- n,s positive integers
- $U = <u_1, \ldots, u_m>$ a seq of integers
- U is called an (n,s) DS sequence if
  - $\forall i \; 1 \le i \le n$
  - $\forall i < m, u_i \neq u_{i+1}$
  - there do not exist s+2 indices $i1<i2<\ldots<i_{s+2}$ so that $u_{i1}=u_{i3}=\ldots=j$ and $u_{i2}=u_{i4}=\ldots=k$ for two distinct numbers $1 \le j,k \le n$

(we call it DS Seq of order s on n symbols, or DS(n,s) for short)

- $\lambda_s(n) = \max \{ |U| \mid U \text{ is a } DS(n,s) \}$

Lower envelopes

- $F=\{f_1,f_2,\ldots,f_n\}$:a set of continuous function defined over an interval I, every pair intersect in at most s points
- m: the minimal number of subintervals such that over each of them the lower envelopes is uniquely defined by a function of F with index u_i
- let $U(f_1,f_2,\ldots,f_n) = \langle u_1,u_2,\ldots,u_m \rangle$

DS sequences and envelopes

- claim 1: $U(f_1,f_2,\ldots,f_n)$ is a DS(n,s)
- Claim 1': for every DS(n,s) U, there exist functions $g_1,\ldots,g_n$ such that $U(g_1,\ldots,g_n) = U$

for functions f_i partially denied over I

- claim 2: $U(f_1,f_2,\ldots,f_n)$ is a DS(n,s+2)
- Claim 2': for every DS(n,s+2) U, there exist partially defined functions $g_1,\ldots,g_n$ such that $U(g_1,\ldots,g_n) = U$

The maximum complexity of envelopes

namely the maximum value of $\lambda_s(n\ )$ (or $\lambda_{s+2}(n\ )$)

$\lambda_1(n\ ) = n$

$\lambda_2(n\ ) = 2n-1$

$\lambda_3(n\ )$ = take I: $O(n \log n)$

$\lambda_s(n\ )$ = take I: $O(n \log^* n)$

The maximum complexity of envelopes

namely the maximum value of $\lambda_s(n\ )$ (or $\lambda_{s+2}(n\ )$)

$\lambda_1(n\ ) = n$

$\lambda_2(n\ ) = 2n-1$

$\lambda_3(n\ ) = \theta(n\alpha(n))$

$\lambda_s(n\ )$ = see references

Constructing envelopes

- divide & conquer
- Hershberger's improvement

The complexity of a single face

- with some care show that the appearance of curves along a CCB of the marked face constitute a DS sequence
- The complexity of a face in an arrg where each pair of curves intersect at most s times is
  - $O(\lambda_s(n))$ for unbounded curves,
  - $O(\lambda_{s+2}(n))$ for bounded curves,
  - $O(\lambda_{s+1}(n))$ for curves bounded on one side

Constructing a single face

- deterministic algorithm, $O(\lambda_{s+2}(n)\log^2 n)$ time
- randomized algorithms, expected

  $O(\lambda_{s+2}(n)\log n)$ time
- $\lambda_{s+2}(n)$ replaced by $\lambda_s(n)$ for unbounded curves

- for bounded curves: the complexity and construction of the zone follows

Operations on arrangements

- traversals
- point location
- overlay
  - Boolean operations

What of it is in CGAL arrangements?

almost everything:

- different families of curves (in the form of traits classes)
- Dcel and traversals
- point location: simple, walk-along-a-line, RIC-based, landmarks
- incremental and zone construction
- vertical decomposition
- envelopes
- overlay
- Boolean operations
- all extended to a families of parametric surfaces

References

- [Sharir -Agarwal '95]
  Davenport-Schinzel Sequences and Their Geometric Applications, Cambridge U Press
- [Halperin '04]
  Arrangements (Ch 24), in *CRC Handbook on Discrete and Computational Geometry*
- [Agarwal-Sharir '00]
  Chapters 1&2 in *NH Handbook of Computational Geometry*

THE END