Construction of General Two-Dimensional Voronoi Diagrams via Divide and Conquer Algorithm of Envelopes

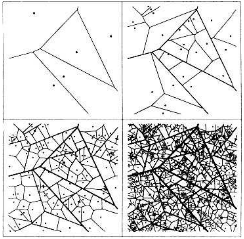
Ophir Setter

Tel-Aviv University, Israel

Applied aspects of Computational Geometry, June 2009

Voronoi Diagrams

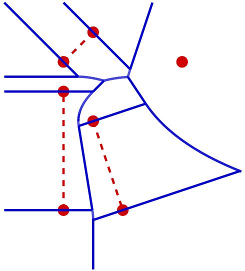
- Given *n* objects (Voronoi sites) in some space (e.g., $\mathbb{R}^d$, $\mathbb{S}^d$) and a distance function ρ
- The Voronoi Diagram subdivides the space into cells
- Each cell consists of points that are closer to one particular site than to any other site
- Variants include different:
  - Classes of sites
  - Embedding spaces
  - Distance functions



Fractals from Voronoi diagrams
http://www.righto.com/fractals/vor.html

Voronoi Diagrams

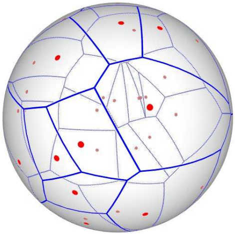
- Given *n* objects (Voronoi sites) in some space (e.g., $\mathbb{R}^d$, $\mathbb{S}^d$) and a distance function ρ
- The Voronoi Diagram subdivides the space into cells
- Each cell consists of points that are closer to one particular site than to any other site
- Variants include different:
  - Classes of sites
  - Embedding spaces
  - Distance functions



Voronoi diagram of segements

Voronoi Diagrams

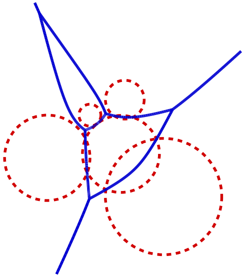
- Given *n* objects (Voronoi sites) in some space (e.g., $\mathbb{R}^d$, $\mathbb{S}^d$) and a distance function ρ
- The Voronoi Diagram subdivides the space into cells
- Each cell consists of points that are closer to one particular site than to any other site
- Variants include different:
  - Classes of sites
  - Embedding spaces
  - Distance functions



Voronoi diagram on the sphere

Voronoi Diagrams

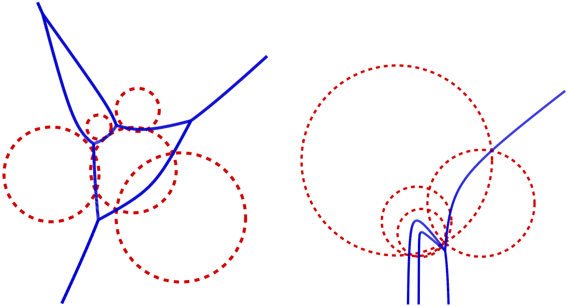
- Given n objects (Voronoi sites) in some space (e.g., $\mathbb{R}^d$, $\mathbb{S}^d$) and a distance function ρ
- The Voronoi Diagram subdivides the space into cells
- Each cell consists of points that are closer to one particular site than to any other site
- Variants include different:
  - Classes of sites
  - Embedding spaces
  - Distance functions



Apollonius diagram

Sample of types of Voronoi diagrams

Apollonius Diagram — Diagram of Disks



Apollonius diagram

Given a set $D = \{d_1, \ldots, d_n\}$ of disks with respective centers p_i and radii r_i, the diagram is defined by the following distance function:

$$\rho(x, d_i) = ||x - p_i|| - r_i$$

Sample of types of Voronoi diagrams

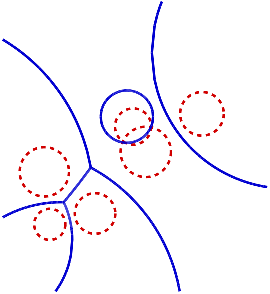
Weighted Diagrams

- Apollonius diagrams are actually additively-weighted Voronoi diagram

Sample of types of Voronoi diagrams

Weighted Diagrams

- Apollonius diagrams are actually additively-weighted Voronoi diagram
- Another type of Voronoi diagrams are the multiplicatively-weighted Voronoi diagrams, or their generalization *Möbius diagrams*



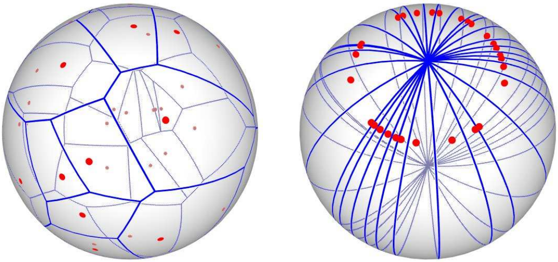
Möbius diagram

Let w_i be a *Möbius site* defined by a triple (p_i, λ_i, μ_i) where $p_i \in \mathbb{R}^2$ and $\lambda_i, \mu_i \in \mathbb{R}$. The diagram is defined by the following distance function:

$$\rho(x, w_i) = \lambda_i (x - p_i)^2 - \mu_i$$

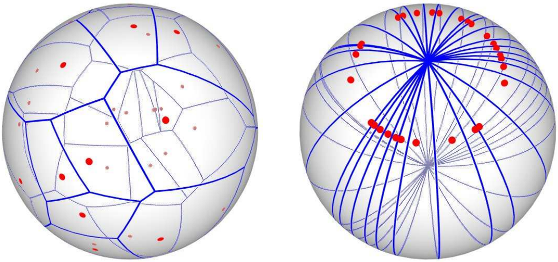
Sample of types of Voronoi diagrams

On the Sphere and others



Sample of types of Voronoi diagrams

On the Sphere and others

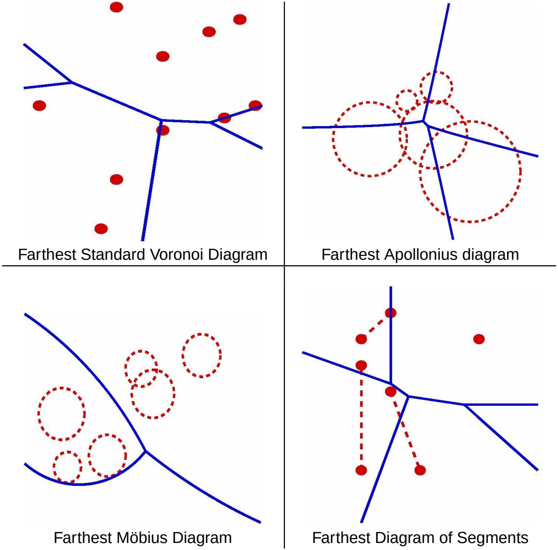


Other types include:

- Taxi-driver (Manhattan) distance
- Moscow (Karlsruhe) distance

Sample of types of Voronoi diagrams

Farthest-site Voronoi diagrams



Farthest Standard Voronoi Diagram

Farthest Apollonius diagram

Farthest Möbius Diagram

Farthest Diagram of Segments

Applications

- Knuth's Post-Office problem
- Largest empty circle and other proximity problems.

Applications

- Knuth's Post-Office problem
- Largest empty circle and other proximity problems.
- Find the most similar object in a database
- Motion planning — Finding clear paths

Applications

- Knuth's Post-Office problem
- Largest empty circle and other proximity problems.
- Find the most similar object in a database
- Motion planning — Finding clear paths
- Epidemiology
  Voronoi diagrams were used to analyze the 1854 cholera epidemic in London. A correlation between deaths and proximity to a particular water pump was discovered

Applications

- Knuth's Post-Office problem
- Largest empty circle and other proximity problems.
- Find the most similar object in a database
- Motion planning — Finding clear paths
- Epidemiology
  Voronoi diagrams were used to analyze the 1854 cholera epidemic in London. A correlation between deaths and proximity to a particular water pump was discovered
- Minimum-width annulus — to come...
- Astronomy, Climatology, Computer Graphics, Chemistry, Architecture, Geology, Zoology, and more

Outline

Lower Envelopes

Definition

Given a set of bivariate functions $S = \{s_1, \ldots, s_n\}$, their lower envelope is defined to be their pointwise minimum:

$$\Psi(x, y) = \min_{1 \le i \le n} s_i(x, y)$$

Lower Envelopes

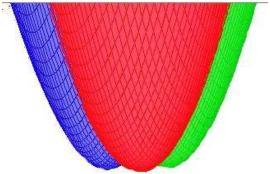
Definition

Given a set of bivariate functions $S = \{s_1, \ldots, s_n\}$, their lower envelope is defined to be their pointwise minimum:

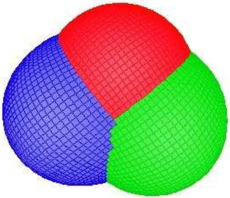
$$\Psi(x, y) = \min_{1 \le i \le n} s_i(x, y)$$

Corollary

Voronoi diagrams are the minimization diagrams (planar projections of the lower envelopes) of the distance functions from each site [Edelsbrunner-Seidel '86]



Distance functions are paraboloids



Looking from bottom gives us the Voronoi diagram

Abstract Voronoi Diagrams

- The distance function only tells us the distance from a point
- Voronoi diagrams can be equivalently defined in terms of their bisectors
- The bisector $B(o_i, o_j)$ of two sites is the locus of all points that have an equal distance to both sites

Abstract Voronoi Diagrams

- The distance function only tells us the distance from a point
- Voronoi diagrams can be equivalently defined in terms of their bisectors
- The bisector $B(o_i, o_j)$ of two sites is the locus of all points that have an equal distance to both sites

Definition

The abstract Voronoi diagram is defined in terms of bisector and dominance regions (partial definition)

$$Reg(o_i, O) = \bigcap_{o_j \in O, j \neq i} Reg(o_i, o_j)$$

Abstract Voronoi Diagrams

- The distance function only tells us the distance from a point
- Voronoi diagrams can be equivalently defined in terms of their bisectors
- The bisector $B(o_i, o_j)$ of two sites is the locus of all points that have an equal distance to both sites

Definition

The abstract Voronoi diagram is defined in terms of bisector and dominance regions (partial definition)

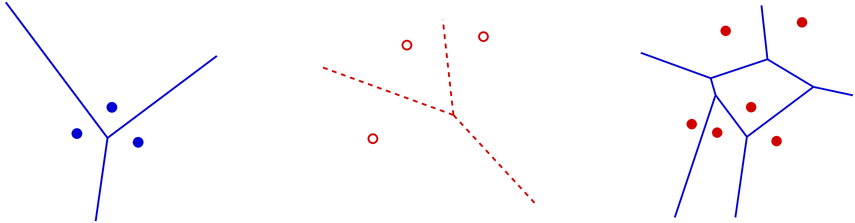
$$Reg(o_i, O) = \bigcap_{o_j \in O, j \neq i} Reg(o_i, o_j)$$

- The bisector is the projected intersection of the distance functions
- What happens if you invert the dominance regions?

The Divide-and-Conquer Algorithm

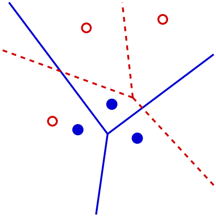
Let S be a set of n sites

1. Partition S into two disjoint subsets S_1 and S_2 of equal size
2. Construct $Vor_\rho(S_1)$ and $Vor_\rho(S_2)$ recursively
3. Merge the two Voronoi diagrams to obtain $Vor_\rho(S)$



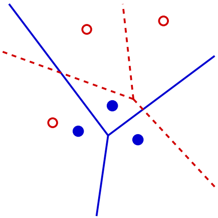
The Merging Step

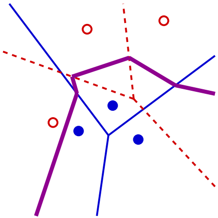
1. Overlay $Vor_\rho(S_1)$ and $Vor_\rho(S_2)$



The Merging Step

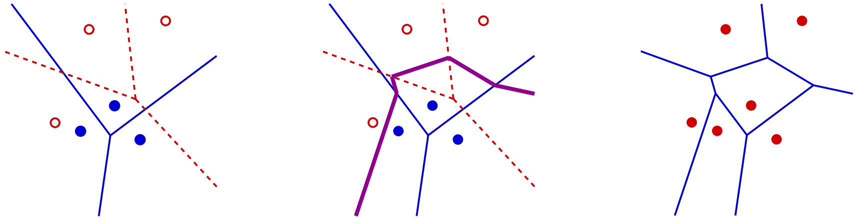
1. Overlay $Vor_\rho(S_1)$ and $Vor_\rho(S_2)$
2. Partition each face to points closer to a site in S_1 and points closer to a site in S_2
3. Label feature of the refined overlay with the sites nearest to it





The Merging Step

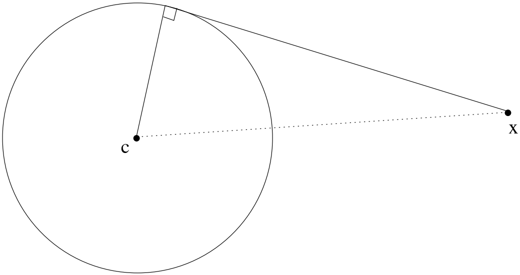
1. Overlay $Vor_\rho(S_1)$ and $Vor_\rho(S_2)$
2. Partition each face to points closer to a site in S_1 and points closer to a site in S_2
3. Label feature of the refined overlay with the sites nearest to it
4. Remove redundant features



Power Distance

The power distance between a circle and a point in the plane:

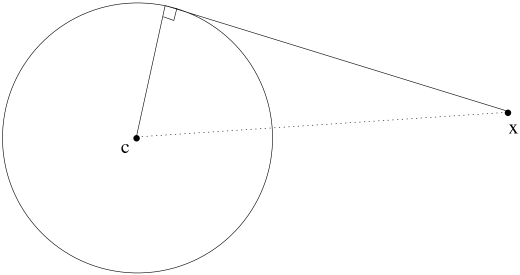
$$\rho(x, d_{c,r}) = ||x - c||^2 - r^2$$



Power Distance

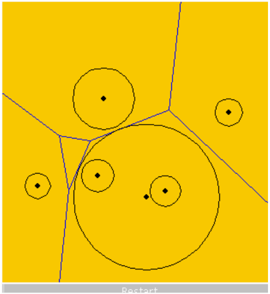
The power distance between a circle and a point in the plane:

$$\rho(x, d_{c,r}) = ||x - c||^2 - r^2$$

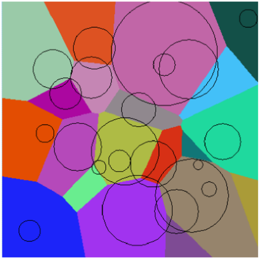


- Approximates the Euclidean distance function, e.g., inside the circle the distance is negative, outside — positive
- Bisector passes through the intersection points of two circles

Power diagram

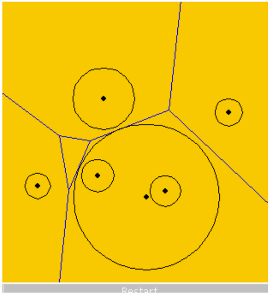


groups.csail.mit.edu

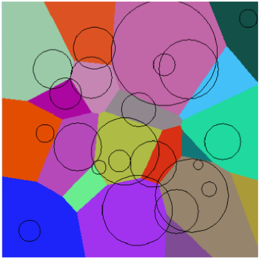


www.jaist.ac.jp

Power diagram



groups.csail.mit.edu



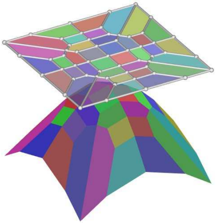
www.jaist.ac.jp

Applications:

- Determine whether a point is inside the union of n circles
- Find the boundary of the union of n circles
- Classify n circles into connected components
  - Mobile phones — can establish a line only if in the same connected component

Constructing the Power Diagram

$$\rho(x, d_i) = ||x - c_i||^2 - r_i^2$$



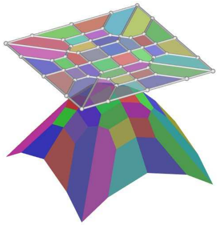
The Voronoi diagram of 36 points

Constructing the Power Diagram

$$\rho(x, d_i) = ||x - c_i||^2 - r_i^2$$

$$f_i(x) = x^2 - 2xc_i + c_i^2 - r_i^2$$

$$\pi_i : -2xc_i + c_i^2 - r_i^2$$



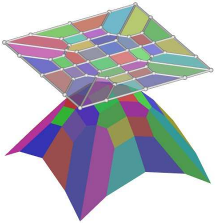
The Voronoi diagram of 36 points

Constructing the Power Diagram

$$\rho(x, d_i) = ||x - c_i||^2 - r_i^2$$

$$f_i(x) = x^2 - 2xc_i + c_i^2 - r_i^2$$

$$\pi_i : -2xc_i + c_i^2 - r_i^2$$



The Voronoi diagram of 36 points

Corollary

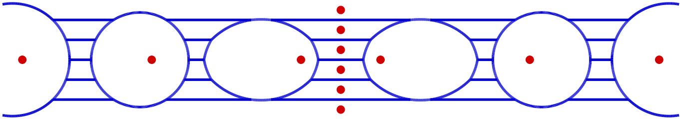
The bisectors of the power diagram of circles in the plane are lines. We can compute the power diagram of circles by computing the lower envelope of planes. In fact, all diagrams whose bisectors are lines are power diagrams.

Multiplicatively-weighted Voronoi diagram

- Multiplicatively-weighted Voronoi diagram of sites $s_i = (p_i, w_i)$ is defined by:

$$\rho(x, s_i) = w_i ||x - p_i||^2$$

- Useful, for example, in modelling the growth of crystals
- The complexity of the diagram could be quadratic

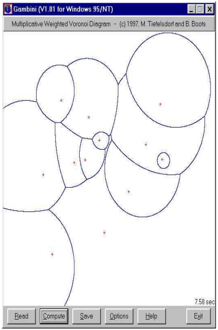


Möbius diagram

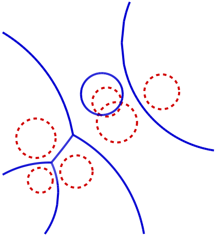
- The Möbius diagram is a generalization of the multiplicatively-weighted Voronoi diagram and is defined by:

$$\rho(x, s_i) = w_i ||x - p_i||^2 + v_i$$

- The bisectors of the diagram are circles
- Every Voronoi diagram with circles as bisectors is Möbius diagram



info.wlu.ca

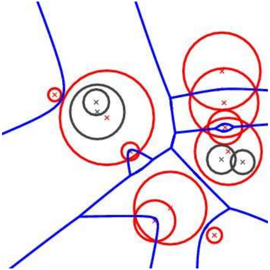

Möbius Voronoi diagram

Apollonius Diagrams

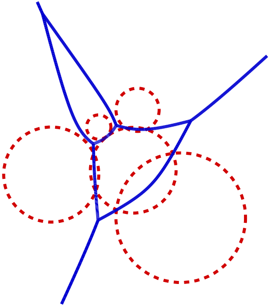
- The Apollonius diagram of disks $d_i = (p_i, r_i)$ is defined by the following distance function:

$$\rho(x, d_i) = ||x - p_i|| - r_i$$

- Negative distance inside the disks
- Bisectors are hyperbolic arcs
- Useful for ...

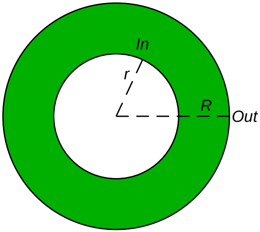


www.cgal.org



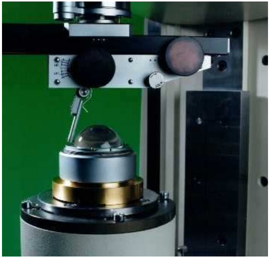
Application: Minimum-Width Annulus

- An annulus is the bounded area between two concentric circles
- The width of an annulus is the difference between its radii R and r
- **Goal:** given a set S of objects (points, segments, etc.) find an annulus of minimum width containing the objects (MWA)



Tolerancing Metrology

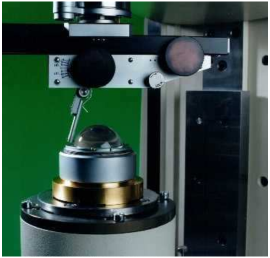
- Roundness is the measure of sharpness of a particle's edges and corners
- In mechanical design there is a need to assess the roundness error of a manufactured object (to see it was manufactured correctly)
- 4 ANSI & ISO methods for round manufactured object assessment:
  - Minimum circumscribed circle (MCC)
  - Maximum inscribed circle (MIC)
  - Least square circle (LSC)
  - Minimum Zone circle (MZC)



www.npl.co.uk/server.php

Tolerancing Metrology

- Roundness is the measure of sharpness of a particle's edges and corners
- In mechanical design there is a need to assess the roundness error of a manufactured object (to see it was manufactured correctly)
- 4 ANSI & ISO methods for round manufactured object assessment:
  - Minimum circumscribed circle (MCC)
  - Maximum inscribed circle (MIC)
  - Least square circle (LSC)
  - Minimum Zone circle (MZC)



www.npl.co.uk/server.php

Minimum-Width Annulus

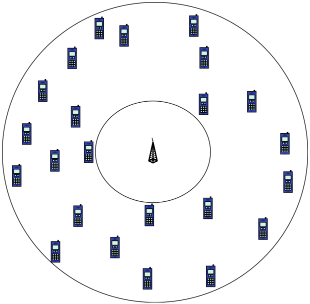
Facility Location

- Place a new cell tower given the location of clients' cell phones
- Cell site has both desirable properties (supplement of service) and obnoxious properties (health)

cgm.cs.mcgill.ca/ athens/cs507/Projects/2004/Emory-Merryman

Facility Location

- Place a new cell tower given the location of clients' cell phones
- Cell site has both desirable properties (supplement of service) and obnoxious properties (health)
- A good location is the center of the minimum-width annulus



cgm.cs.mcgill.ca/ athens/cs507/Projects/2004/Emory-Merryman

Related Work

Points:

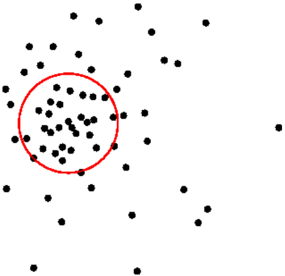
- Rivlin [Riv79] and afterwards Smid and Janardan [SJ99] proved the theorem for points (following slides)
- Algorithms based on this theorem and Voronoi Diagrams introduced by Ebara [EFNN89] and by Roy and Zhang [RZ92]

Variants:

- Constrained MWA by de Berg et al. [dBBB$^+$98] — useful enforcing different restrictions on roundness
- MWA of a polygon by Le and Lee [LL91]
  - Not identical to the MWA of segments
  - Controversy on the right algorithm for roundness

Related Work — cont.

- Special Sub-quadratic Algorithms:
  - Linear and $O(n \log n)$ algorithms for special cases with more information (Garcia-Lopez et al. '98, Swanson et al. '93, Devillers and Ramos '02)
- Best known randomized algorithm (with expected running time of $O(n^{3/2+\varepsilon})$) is achieved by solving a 3D width-like problem after lifting the points by $(x, y) \mapsto (x, y, x^2 + y^2)$
  - A circle $\mapsto$ a plane
  - 2 concentric circles $\mapsto$ 2 parallel planes
- Using coresets Chan [Cha06] presented an $(1 + \varepsilon)$-factor approximation algorithm



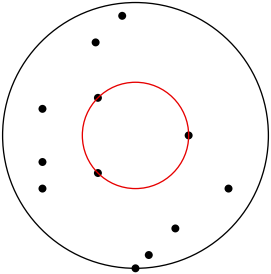
www.almaden.ibm.com/u/kclarkson/sga/t/t.xml

The Connection to Voronoi Diagrams

MWA does not always exist — This is possible in case of the width of the points is smaller than any width of any annulus (collinear points). 3 cases:

The Connection to Voronoi Diagrams

MWA does not always exist — This is possible in case of the width of the points is smaller than any width of any annulus (collinear points).
3 cases:
Inner circle touches 3 points — center is a nearest Voronoi vertex

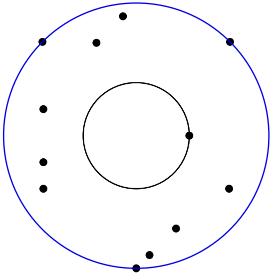


The Connection to Voronoi Diagrams

MWA does not always exist — This is possible in case of the width of the points is smaller than any width of any annulus (collinear points).

3 cases:

Outer circle touches 3 points — center is a farthest Voronoi vertex

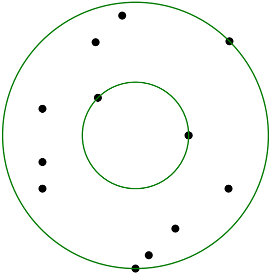


The Connection to Voronoi Diagrams

MWA does not always exist — This is possible in case of the width of the points is smaller than any width of any annulus (collinear points).

3 cases:

Both inner and outer circles touches $\geq$ 2 points — center is an intersection point between the diagrams (on edges of both diagrams)

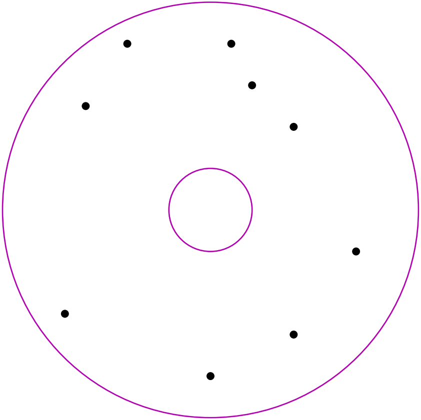


With points if MWA exist only last case is possible

Proof

Observation

Each of the circles touches at least one point

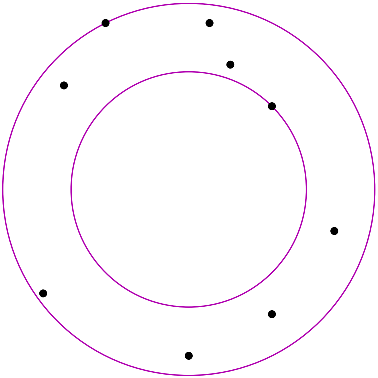


Proof

Observation

Each of the circles touches at least one point

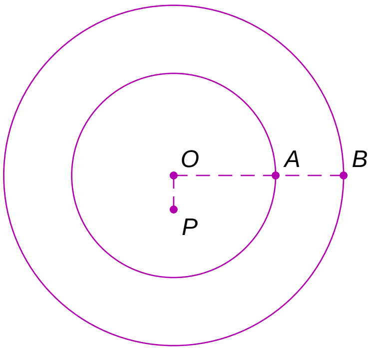
A point fixes an annulus



Proof — cont.

Lemma

At least one of the circles touches more then one point



Proof.

Suppose to contrary. Two cases:

- A is on OB

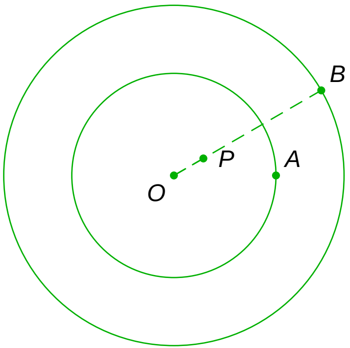
□

$$|PB| < |PA| + |AB|$$
$$r = |AB| > |PB| - |PA|$$

Proof — cont.

Lemma

At least one of the circles touches more then one point



Proof.

Suppose to contrary. Two cases:

- A is on OB
- A is not on OB

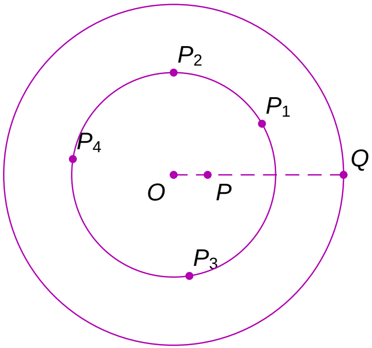
□

$$|OA| < |OP| + |PA|$$
$$r = |OB| - |OA| = |OP| + |PB| - |OA| > |PB| - |PA|$$

Proof — cont.

Theorem

Each circle touches at least 2 points



Proof.

Suppose to contrary.

Case I — *Out* touches one point:

Out touches Q, *In* — P_i

- none of P_i is on OQ

□

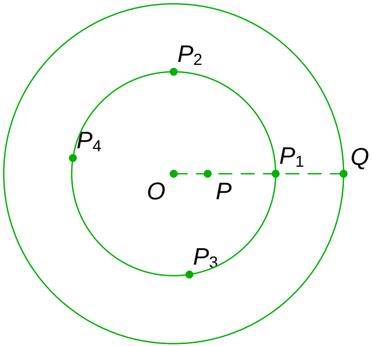
$$|OP| > |OP_i| - |PP_i|$$

$$|OQ| - |OP_i| = |OP| + |PQ| - |OP_i| > |PQ| - |PP_i|$$

Proof — cont.

Theorem

Each circle touches at least 2 points



Proof.

Suppose to contrary.
Case I — *Out* touches one point:
Out touches Q, *In* — P_i

- none of P_i is on OQ
- P_1 is on OQ

□

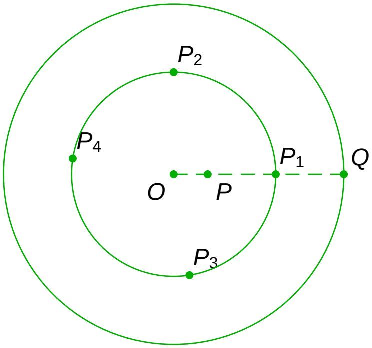
$$|OP| + |PP_i| > |OP_i| = |OP| + |PP_1|$$

Annulus at P is MWA, contradiction

Proof — cont.

Theorem

Each circle touches at least 2 points



Proof.

Suppose to contrary.

Case I — *Out* touches one point:

Out touches Q, *In* — P_i

- none of P_i is on OQ
- P_1 is on OQ

□

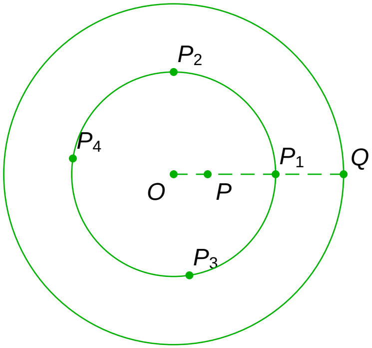
$$|OP| + |PP_i| > |OP_i| = |OP| + |PP_1|$$

Annulus at P is MWA, contradiction

Proof — cont.

Theorem

Each circle touches at least 2 points



Proof.

Suppose to contrary.

Case I — *Out* touches one point:

Out touches Q, *In* — P_i

- none of P_i is on OQ
- P_1 is on OQ

□

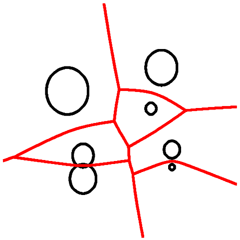
$$|OP| + |PP_i| > |OP_i| = |OP| + |PP_1|$$

Annulus at P is MWA, contradiction

Case II (*In* touches 1 point) — similar

MWA of Disks in the Plane

Nearest Voronoi is replaced by the Apollonius diagram

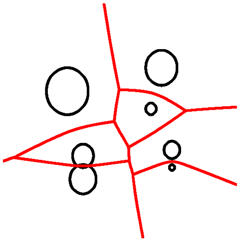


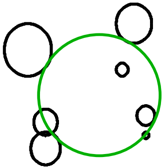
$$\delta(x, d_i) = ||x - c_i|| - r_i$$

MWA of Disks in the Plane

Nearest Voronoi is replaced by the Apollonius diagram

Farthest Apollonius diagram is not good in this case



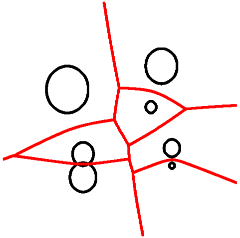


$$\delta(x, d_i) = ||x - c_i|| - r_i$$

We need to consider the farthest point of the disk from a point

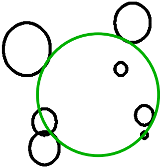
MWA of Disks in the Plane

Nearest Voronoi is replaced by the Apollonius diagram



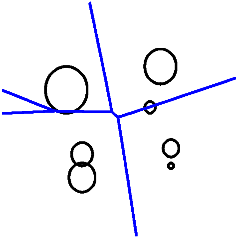
$$\delta(x, d_i) = ||x - c_i|| - r_i$$

Farthest Apollonius diagram is not good in this case



We need to consider the farthest point of the disk from a point

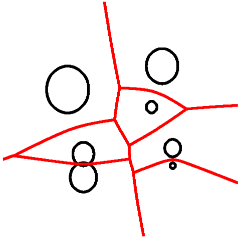
Farthest-Point Farthest-Site VD replaces the farthest VD



$$\delta(x, d_i) = ||x - c_i|| + r_i$$

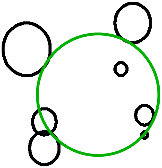
MWA of Disks in the Plane

Nearest Voronoi is replaced by the Apollonius diagram



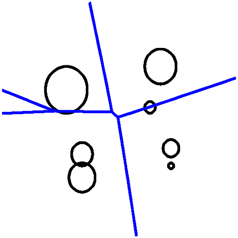
$\delta(x, d_i) = ||x - c_i|| - r_i$

Farthest Apollonius diagram is not good in this case



We need to consider the farthest point of the disk from a point

Farthest-Point Farthest-Site VD replaces the farthest VD

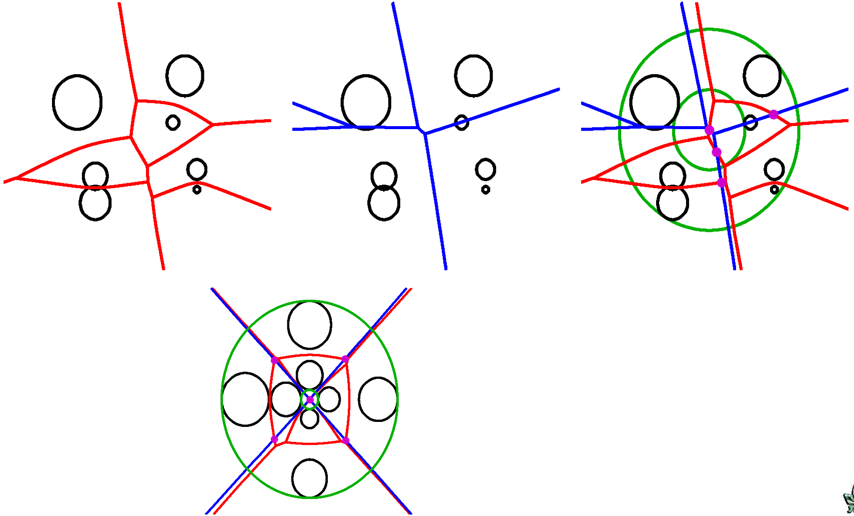


$\delta(x, d_i) = ||x - c_i|| + r_i$

Farthest-point farthest-site is a farthest-site Apollonius with negative radii and was easily produced using our framework

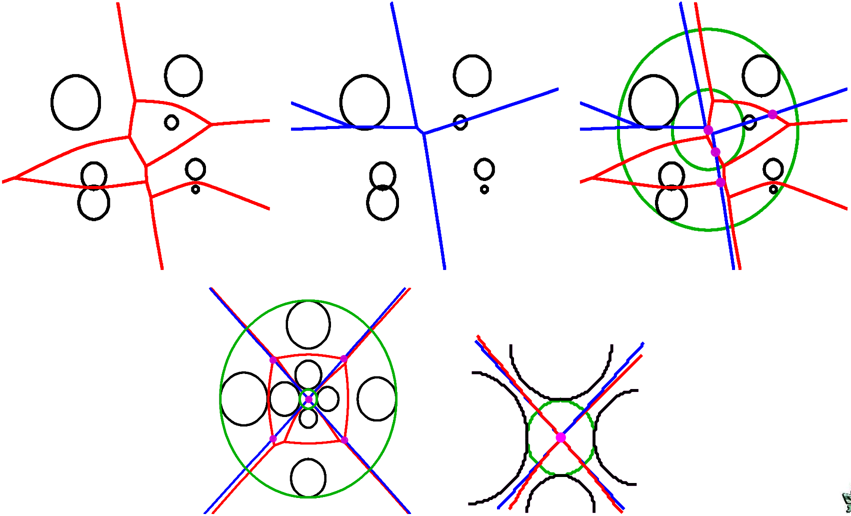
MWA of Disks in the Plane

Cont.



MWA of Disks in the Plane

Cont.



Further information can be found at:

Ophir Setter and Micha Sharir and Dan Halperin.
Constructing two-dimensional Voronoi diagrams via divide-and-conquer of envelopes in space.
In *Proceedings of 6th Annual International Symposium on Voronoi Diagrams in Science and Engineering (ISVD)*, pages 43–52, 2009. Copenhagen, Denmark.

and at the TAU project page:

http://acg.cs.tau.ac.il/projects/internal-projects/vd-via-dc-of-envelopes

Further Reading I

H. Edelsbrunner and R. Seidel.
Voronoi diagrams and arrangements.
Discrete and Computational Geometry, 1:25–44, 1986.

R. Klein.
Concrete and Abstract Voronoi Diagrams, volume 400 of *Lecture Notes in Computer Science*.
Springer, 1989.
ISBN 3-540-52055-4.

M. Meyerovitch.
Robust, Generic, and Efficient Construction of Envelopes of Surfaces in Three-Dimensional Space.
Proc. 14th Annual European Symposium on Algorithms (ESA), 4168:792–803, 2006.

Further Reading II

- J.-D. Boissonnat, C. Wormser, and M. Yvinec.
  Curved voronoi diagrams.
  In J.-D. Boissonnat and M. Teillaud, editors, *Effective Computational Geometry for Curves and Surfaces*. Springer-Verlag, Mathematics and Visualization, 2006.
  URL `http://cgal.inria.fr/Publications/2006/BWY06`.
- A. Fabri and S. Pion.
  A generic lazy evaluation scheme for exact geometric computations.
  In *Proc. 2nd Library-Centric Software Design*, 2006.
  URL `http://cgal.inria.fr/Publications/2006/FP06`.
- S. Pion, M. Teillaud, and C. P. Tsirogiannis.
  Geometric filtering of primitives on circular arcs.
  Technical Report ACS-TR-121, INRIA, 2006.

Further Reading III

- M. Sharir.
  The clarkson-shor technique revisited and extended.
  In *SCG '01: Proceedings of the seventeenth annual symposium on Computational geometry*, pages 252–256, New York, NY, USA, 2001. ACM Press.
  ISBN 1-58113-357-X.
- P. K. Agarwal, O. Schwarzkopf, and M. Sharir.
  The overlay of lower envelopes and its applications.
  Discrete and Computational Geometry, 15:1–13, 1996.
- E. Berberich and M. Meyerovitch.
  Computing envelopes of quadrics.
  In *Proceedings of 23rd European Workshop on Computational Geometry*, pages 235–238, Graz, Austria, March 2007. Technische Universitaet Graz.
  ISBN 978-3-902465-62-7.

Further Reading IV

M. Karavelas and M. Yvinec.
The Voronoi diagram of of planar convex objects.
In *Proc. 11th European Symposium on Algorithms*, volume 2832 of *Lecture Notes in Computer Science*, pages 337–348. Springer-Verlag, 2003.
URL `http://cgal.inria.fr/Publications/2003/KY03b`.

M. I. Karavelas.
A robust and efficient implementation for the segment Voronoi diagram.
In *International Symposium on Voronoi Diagrams in Science and Engineering (VD2004), pages 51–62*

M. I. Karavelas and I. Z. Emiris
Predicates for the planar additively weighted Voronoi diagram. Voronoi diagram.
Technical Report ECG-TR-122201-01, INRIA Sophia-Antipolis

Further Reading V

E. Fogel, D. Halperin, L. Kettner, M. Teillaud, R. Wein and N. Wolpert
Effective Computational Geometry for Curves and Surfaces.
J.-D. Boissonnat and M. Teillaud (eds.)
Chapter 1: *Arrangements*
Springer, 2006.

C. Burnikel, K. Mehlhorn, and S. Schirra.
How to compute the voronoi diagram of line segments: Theoretical and experimental results.
In *ESA '94: Proceedings of the Second Annual European Symposium on Algorithms*, pages 227–239, London, UK, 1994. Springer-Verlag.
ISBN 3-540-58434-X.

Mark de Berg, Prosenjit Bose, David Bremner, Suneeta Ramaswami, and Gordon T. Wilfong.
Computing constrained minimum-width annuli of point sets.
Computer-Aided Design, 30(4):267–275, 1998.

Further Reading VI

Theodore J. Rivlin.
Approximating by circles.
Computing, 21:93–104, 1979.

Michiel H. M. Smid and Ravi Janardan.
On the width and roundness of a set of points in the plane.
International Journal of Computational Geometry and Applications, 9(1):97–108, 1999.

Hiroyuki Ebara, Noriyuki Fukuyama, Hideo Nakano, and Yoshiro Nakanishi.
Roundness algorithms using the Voronoi diagrams.
In *Proceedings of 16th The Canadian Conference on Computational Geometry*, volume 41, 1989.

Utpal Roy and Xuzeng Zhang.
Establishment of a pair of concentric circles with the minimum radial separation for assessing roundness error.
Computer-Aided Design, 24(3):161–168, 1992.

Further Reading VII

Van-Ban Le and Der-Tsai Lee.
Out-of-roundness problem revisited.
IEEE Transactions on Pattern Analysis and Machine Intelligence, 13(3):217–223, 1991.

Arno Eigenwillig and Michael Kerber.
Exact and efficient 2D-arrangements of arbitrary algebraic curves.
In *Proceedings of 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 122–131, Philadelphia, PA, USA, 2008. Society for Industrial and Applied Mathematics (SIAM).

Timothy Moon-Yew Chan.
Faster core-set constructions and data-stream algorithms in fixed dimensions.
Computational Geometry: Theory and Applications, 35(1-2):20–35, 2006.