Algorithmic Robotics and Motion Planning

Translational Motion and Minkowski Sums

Dan Halperin
School of Computer Science
Tel Aviv University

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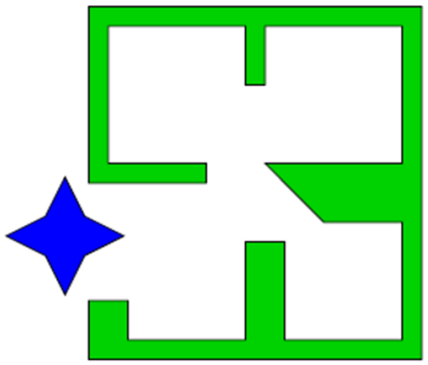
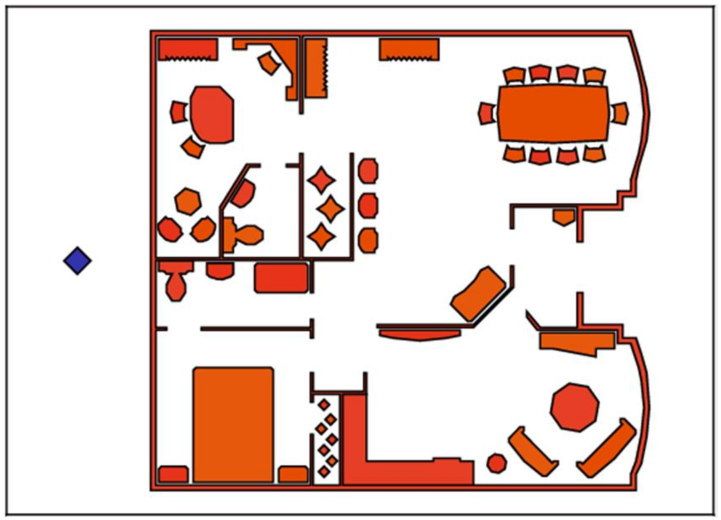
Translational motion of a polygon among polygons in the plane

- Very well understood and having efficient complete implementation (which is not the case for almost any other non-trivial MP problem)
- The structure behind the solutions: Minkowski sums
- A few theoretical problems remain open





Outline

- The combinatorial complexity of polygonal Minkowski sums in the plane
- The connection between motion planning and Minkowski sums
- Algorithms
- Going up to 3D

As time permits:

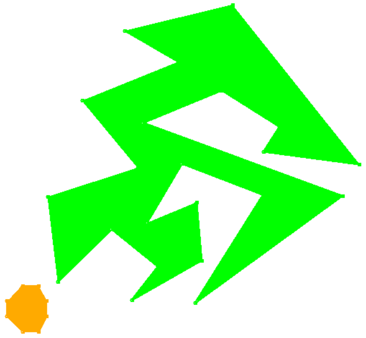
- More applications of Minkowski sums
- Minkowski average

Polygonal Minkowski sums in the plane

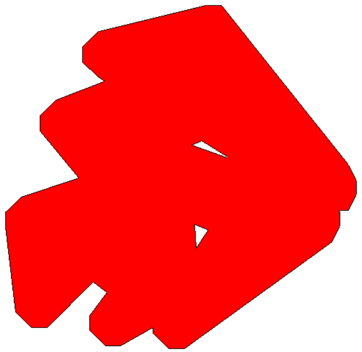
Structure and combinatorial complexity

The Minkowski sum of two sets P and Q in Euclidean space is the result of adding every point in P to every point in Q

$$\{(x_1, y_1)\} \oplus \{(x_2, y_2)\} = \{(x_1 + x_2, y_1 + y_2)\}$$



P, Q



$P \oplus Q$

[wikipedia]

1864 - 1909

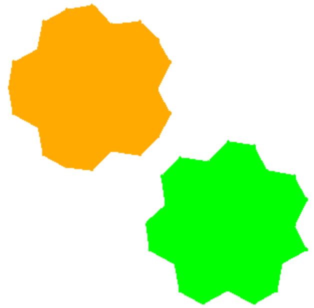
Convex polygons

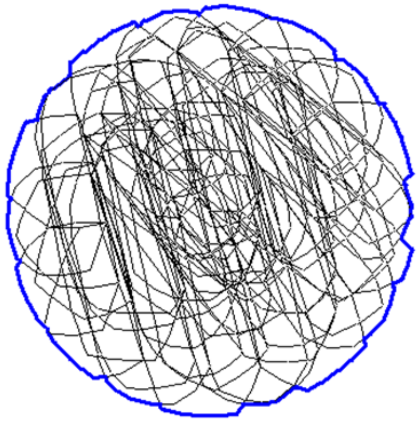
- The farthest point of the sum in any direction is the sum of the farthest points in that direction of the summands
- The sum of convex polygons is a convex polygon
- Given polygons with m and n vertices, the sum has at most $m + n$ vertices

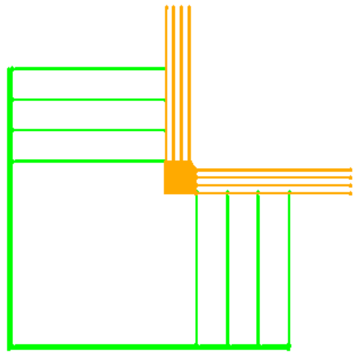
- Later, an interesting property of sums of convex polygons: pseudodiscs

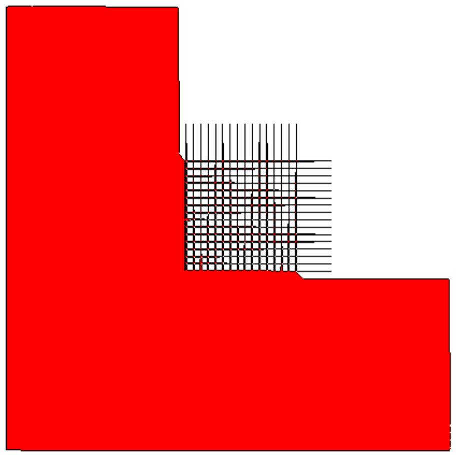
Non-convex polygons

- Triangulate each polygon, construct the union of sums of pairs of triangles
- Arrangements of segments
- Number of segments $O(mn)$
- Maximum complexity $O(m^2n^2)$
- The bound is tight









So far

Given two polygons with m and n vertices

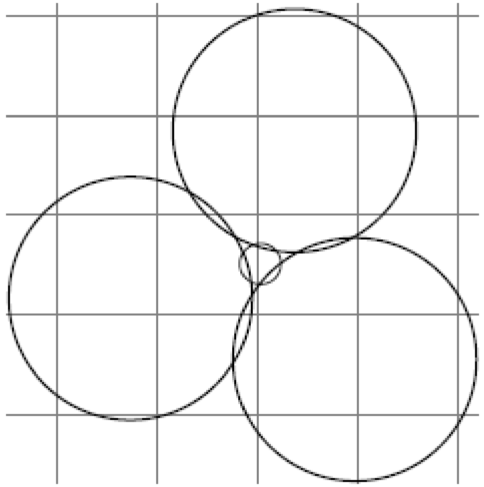
- Convex – convex: $O(m+n)$
- Non-convex – non-convex: $O(m^2n^2)$

How about convex – non-convex?

One of the early surprising results in CG

Reminder: The union of discs

- Given $m > 2$ discs in the plane, the boundary of the union of regions that they enclose contains at most $6m - 12$ intersection points of the arcs
- This bound is tight

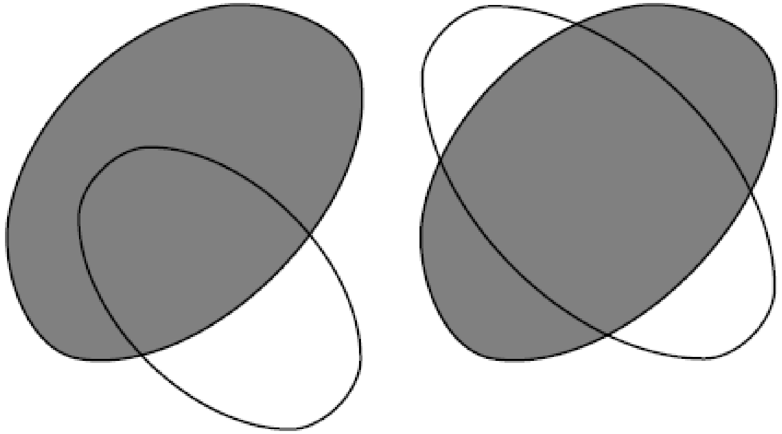


Pseudodiscs

Definition (A pair of pseudodiscs)

A pair of two planar connected point sets o_1 and o_2 is called a pair of pseudodiscs if $\partial o_1 \cap \mathrm{int}(o_2)$ is connected and $\partial o_2 \cap \mathrm{int}(o_1)$ is connected.

- The boundaries ∂o_1 and ∂o_2 intersect in at most two points.



pseudodiscs not pseudodiscs

Convex – non-convex, polygons: The pseudodiscs property

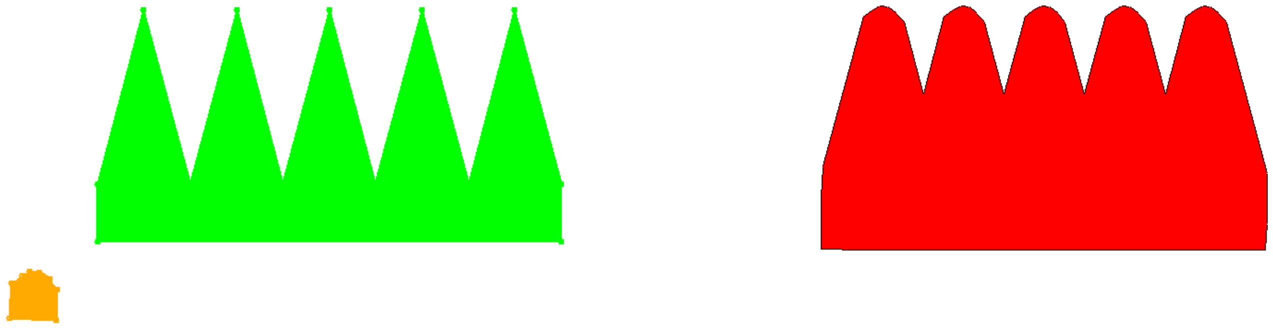
Theorem

Let P and Q be two convex polygons that are interior disjoint, and let R be another convex polygon. The two Minkowski sums $P \oplus R$ and $Q \oplus R$ are pseudodiscs.

Convex – non-convex, polygons: The complexity of the sum

Theorem: the complexity of the Minkowski sum of a convex polygon with m vertices and a simple polygon with n vertices is $O(mn)$

- The bound is tight



General result: The union of pseudodiscs

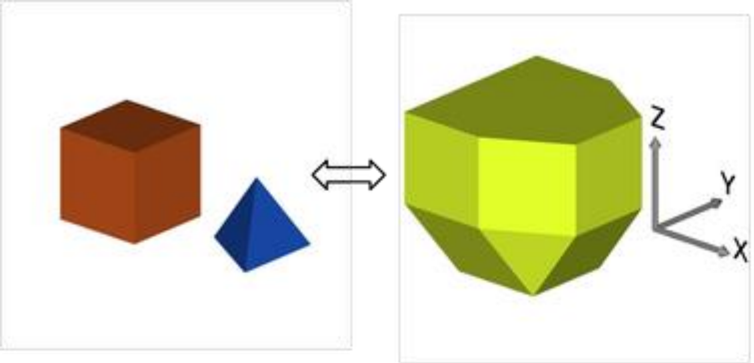
- Given m simple Jordan curves in the plane, each pair of which intersect one another in at most two points, then the boundary of the union of regions that they enclose contains at most $\max(2, 6m - 12)$ intersection points of the curves, and this bound cannot be improved

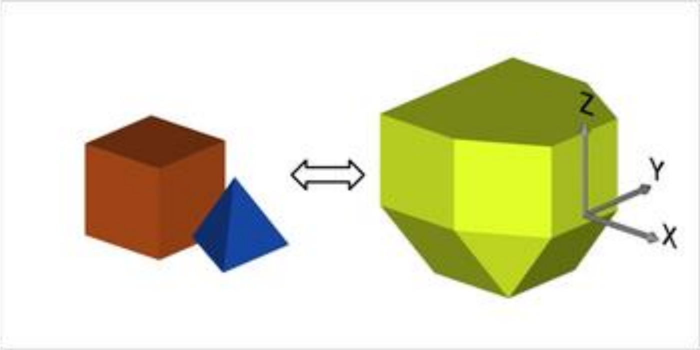
[kedem-Livne-Pach-Sharir ‘86]

Minkowski sums and translational motion planning

Why are Minkowski sums so useful? Here's a major reason:

- Claim: Two sets A and B intersect if and only if the Minkowski sum $A \oplus -B$ contains the origin, where $-B$ is the set B reflected through the origin





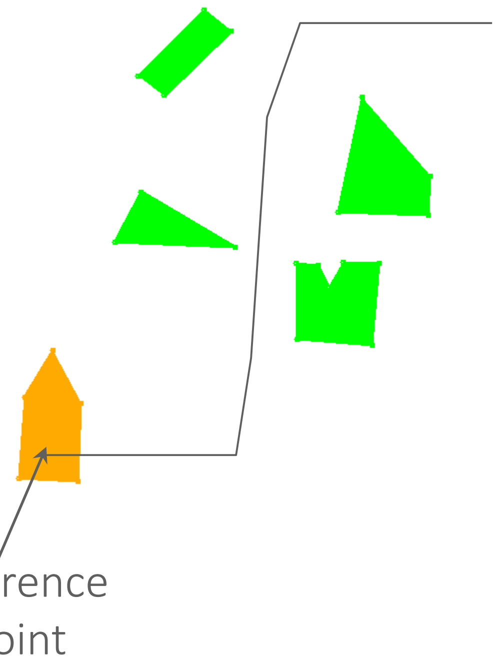
- More generally: $A \cap (B \oplus \{t\}) \neq \emptyset$ iff $t \in A \oplus -B$

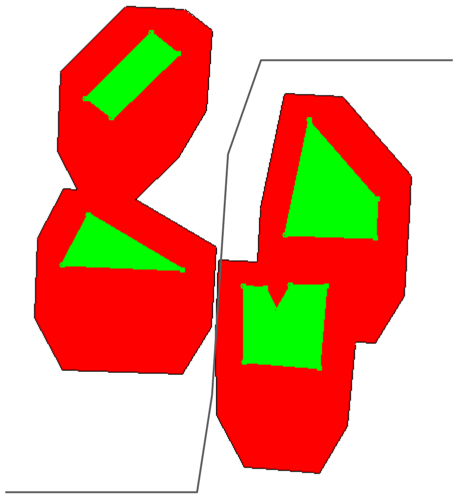
In the plane $-B$ is B rotated by π radians around the origin

Example: motion planning (piano movers)

R - a polygonal object that moves by translation

P - a set of polygonal obstacles





Claim: When translating, R intersects P iff ref(R) is inside $P \oplus -R$

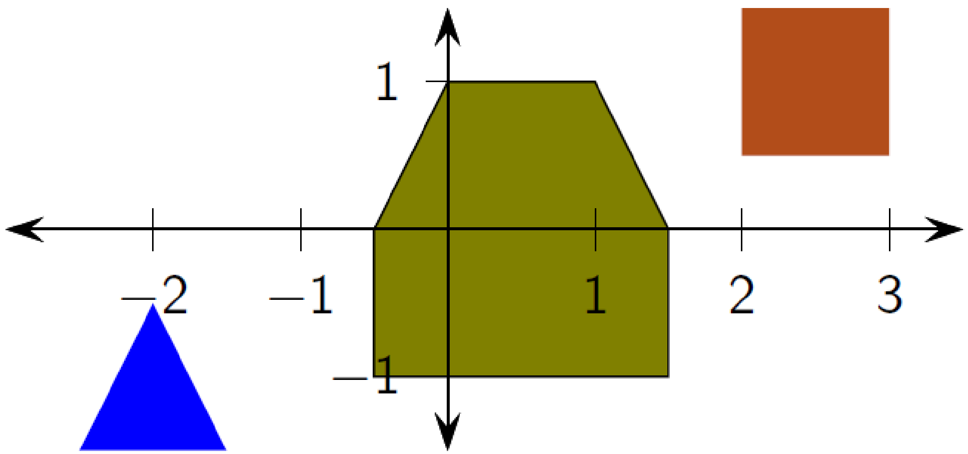
Algorithms

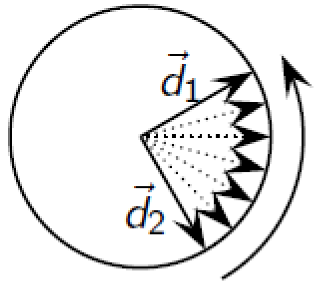
Algorithms

- Convex - convex
- The general case
  - Representation
  - Decomposition
  - The mystery of the construction time
  - Convolution
  - The hole filter
- Convex – non-convex

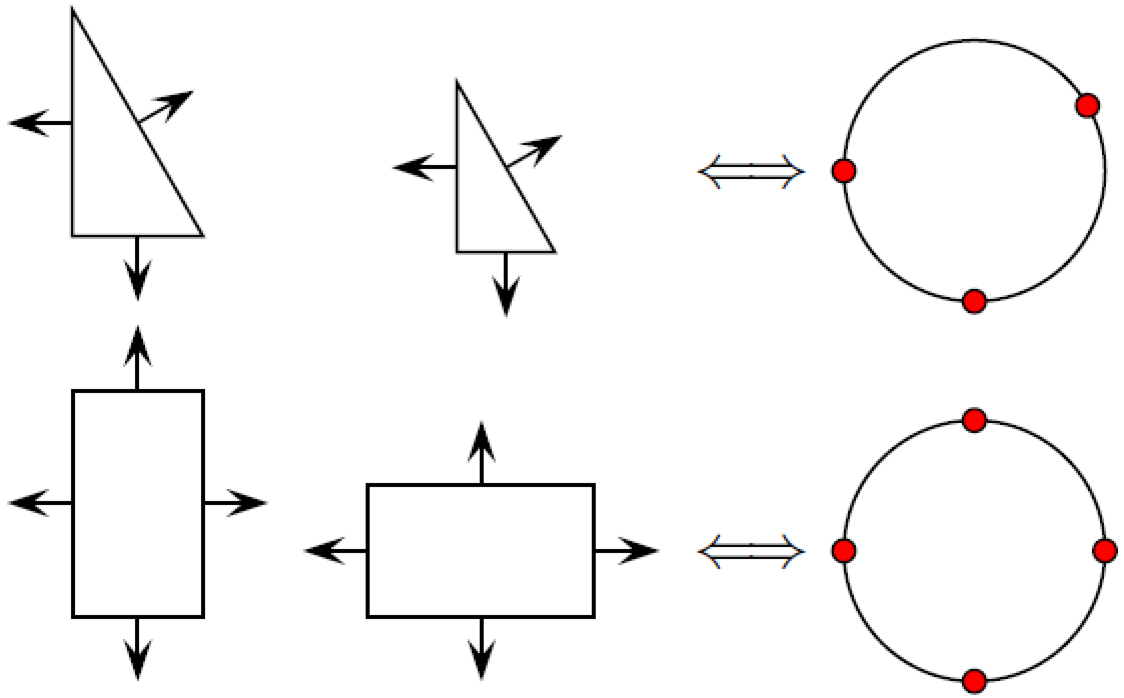
Convex – convex

- Merging of normal diagrams
- O($m + n$)





Directions between $\vec{d}_1$ and $\vec{d}_2$.



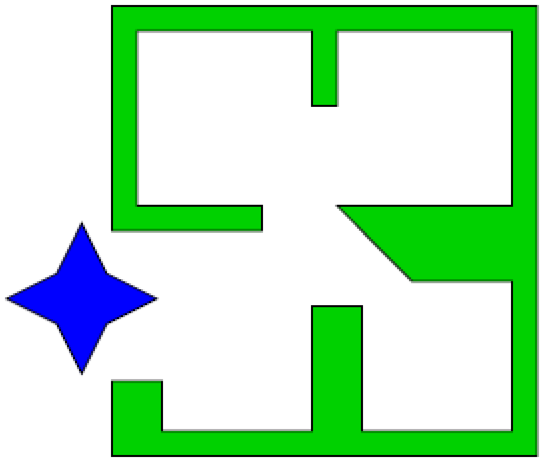
The general case
Construction by decomposition

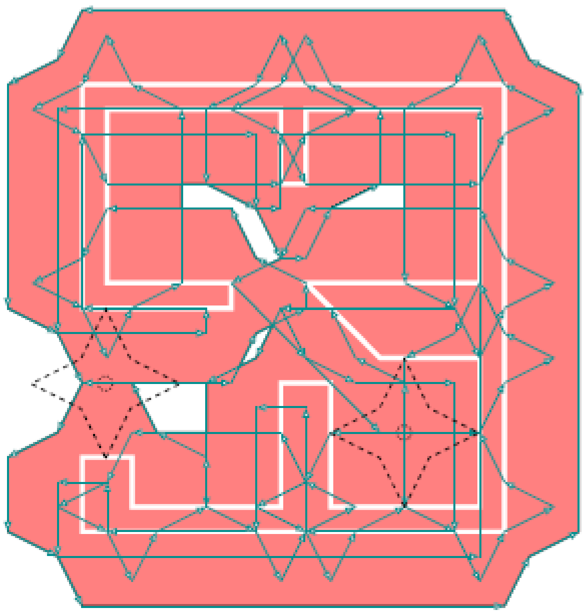
- We alluded to it when we gave the general bound

- Step 1 Decompose P and Q into convex subpolygons $P_1, \dots, Ps$ and $Q_1, \dots, Q_t$
- Step 2 Compute $R_{ij} := P_i \oplus Q_j$ for each pair
- Step 3 Construct the union of those subsums

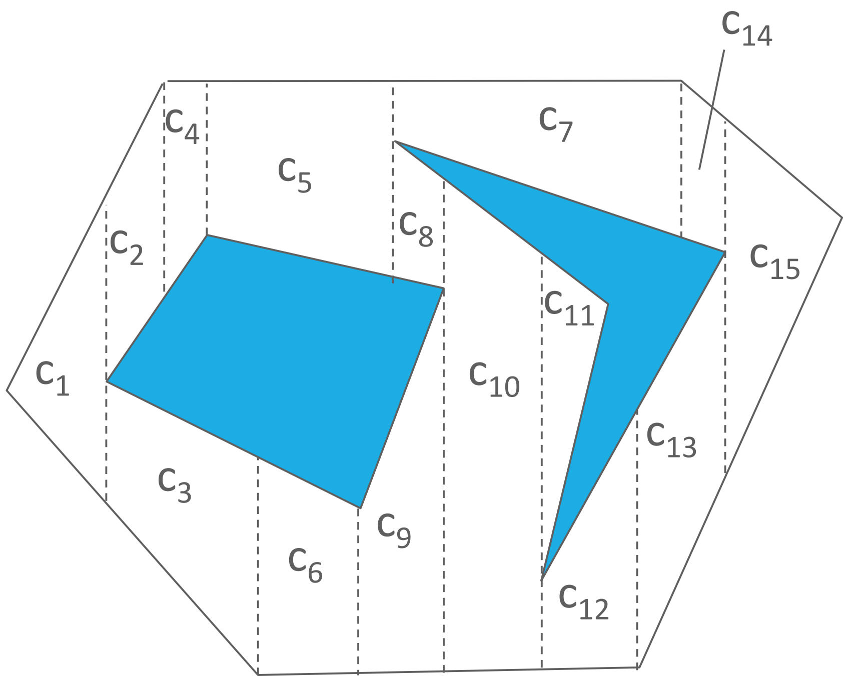
How to represent Minkowski sums?
The language of arrangements

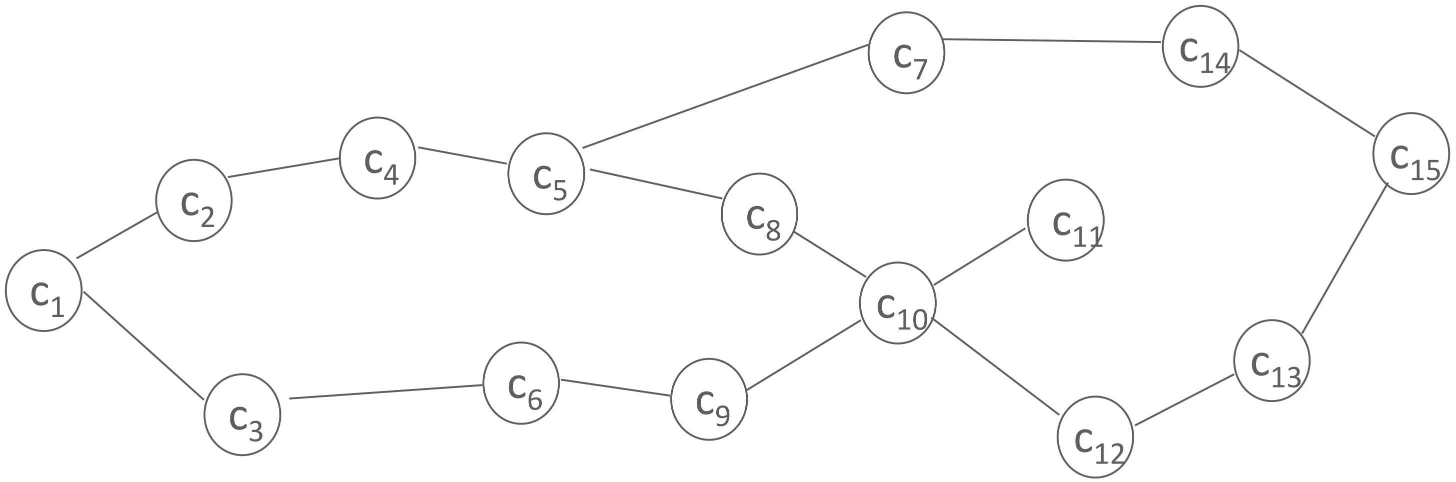
- Much more involved than the convex case
- Should allow for complex topology, holes of any dimension
- Arrangements of curves and surfaces do the job





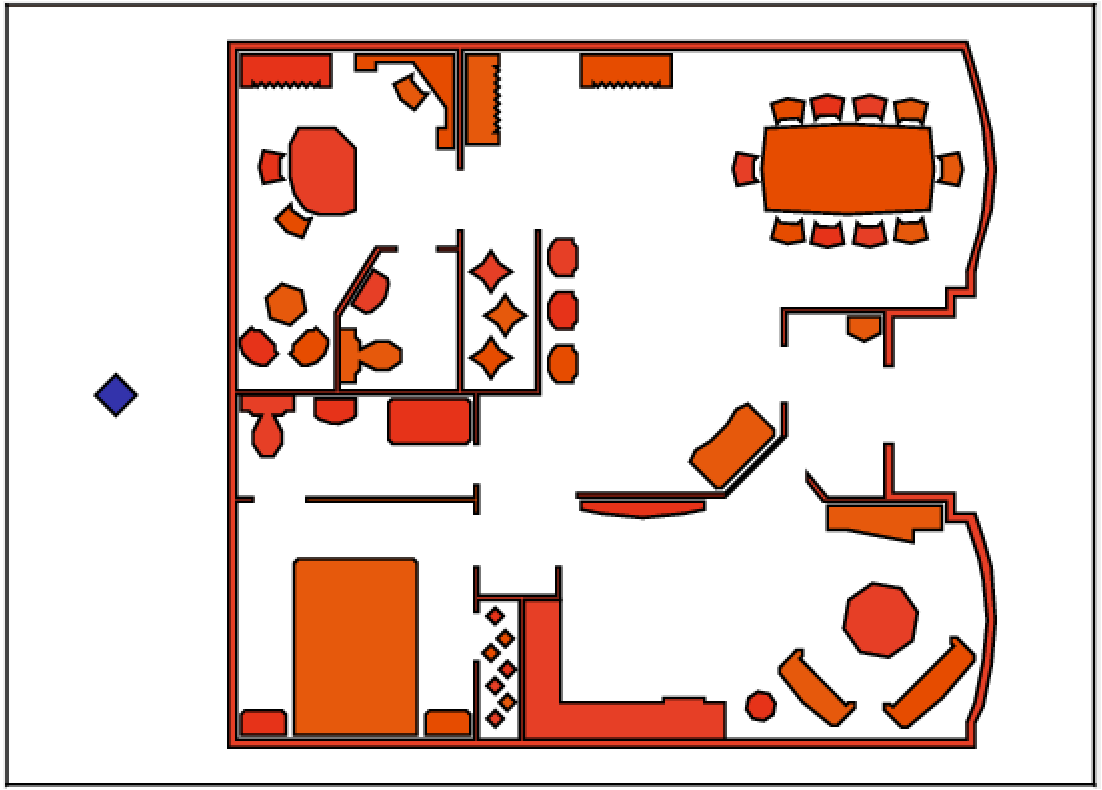
Representation of the free space
Vertical decomposition + connectivity graph

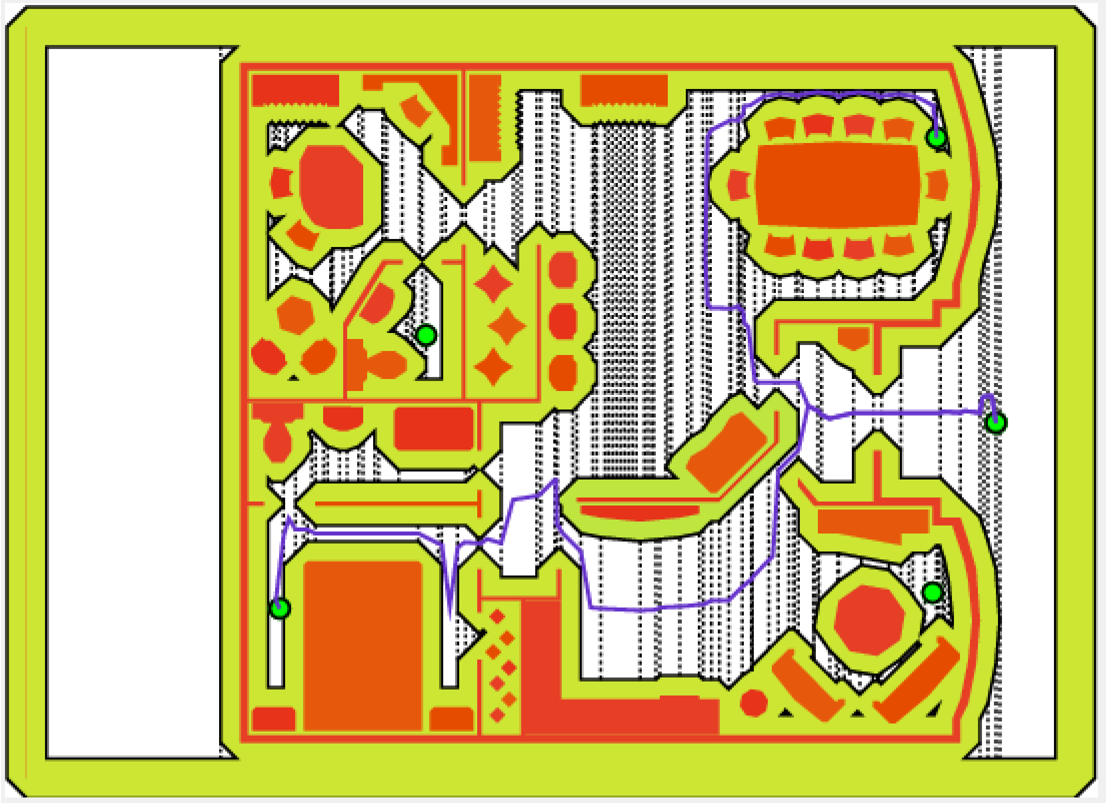




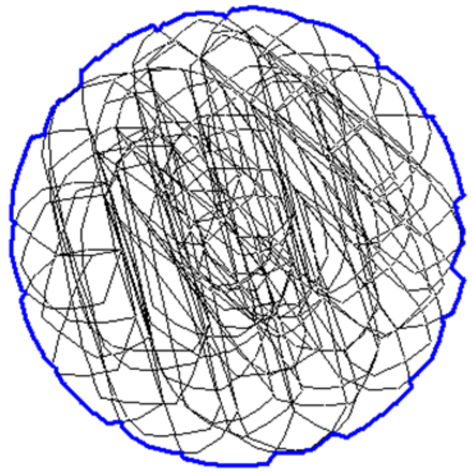
[www.seas.upenn.edu/~jwk/motionPlanning]

Representation of the free space
Example





Constructing the union of the subsums
One possibility: the arrg algorithm



- Let R be the set of all R_{ij} s
- Add all the edges of R into a planar arrangement
- Compute carefully for each face, edge and vertex whether it is inside union
- Time: $O((I + k)\log k)$

  $O(I + k)$ - traversal

  k - number of edges in R

  I - number of intersections among edges of R

The mystery of the construction time

Construction by decomposition, zooming in

- Step 1 Decompose P and Q into convex subparts $P_1, \dots, P_S$ and $Q_1, \dots, Q_t$
- Step 2 Compute $R_{ij} := P_i \oplus Q_j$ for each pair
- Step 3 Construct the union of those subsums

For example: P and Q are polygons with $m + 2$ and $n + 2$ vertices resp.

- Step 1 Decompose P and Q into m and n triangles
- Step 2 Compute the hexagon $R_{ij} := P_i \oplus Q_j$ for each pair
- Step 3 Construct the union of those $4mn$ triangles

Algorithm complexity

- Recall that for arbitrary polygons the maximum complexity is $O(m^2n^2)$
- The output may have size $\Omega(m^2n^2)$
- We know to approach this running time in the worst case
- Can we have a guaranteed output-sensitive algorithm?
- Can we efficiently decide if the Minkowski sum has holes?

Hardness in P

- The curious incident of 3-SUM hard problems

Definition (3SUM)

Given a set S of n integers, are there elements $a, b, c \in S$ such that $a + b + c = 0$?

- Computing the union of a set of triangles is 3SUM hard
- We need to compute the union of $4mn$ triangles
- HOWEVER, our $4mn$ triangles are special
- The mystery remains

[Gajentaan-Overmars ’95], [Grønlund-Pettie ‘14]

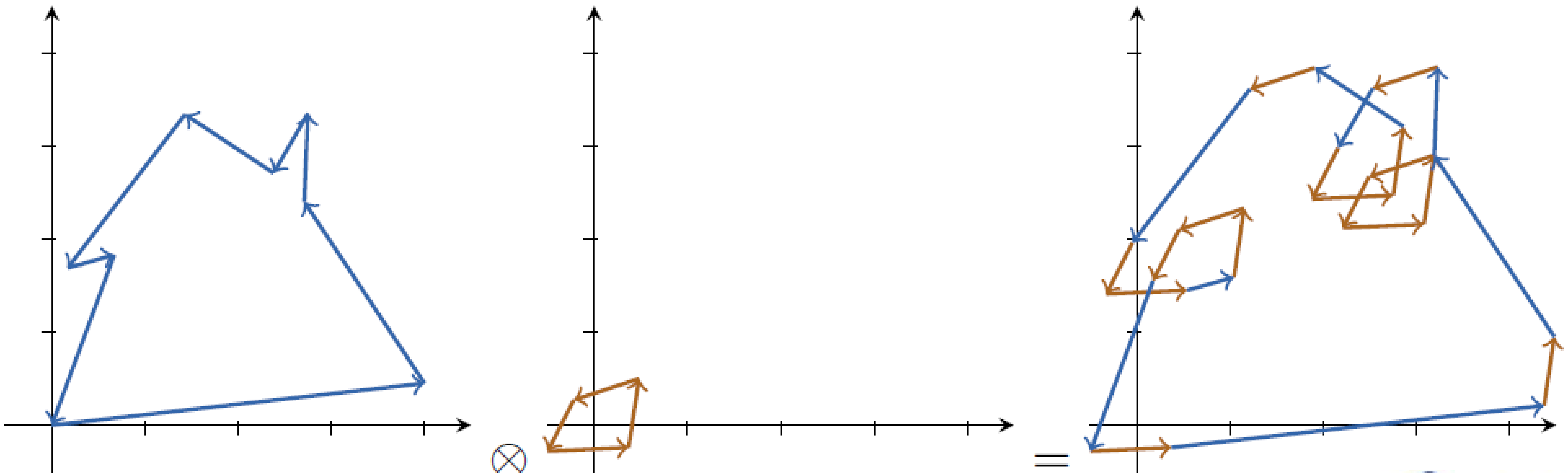
The hole filter

in theory and practice

Convolution

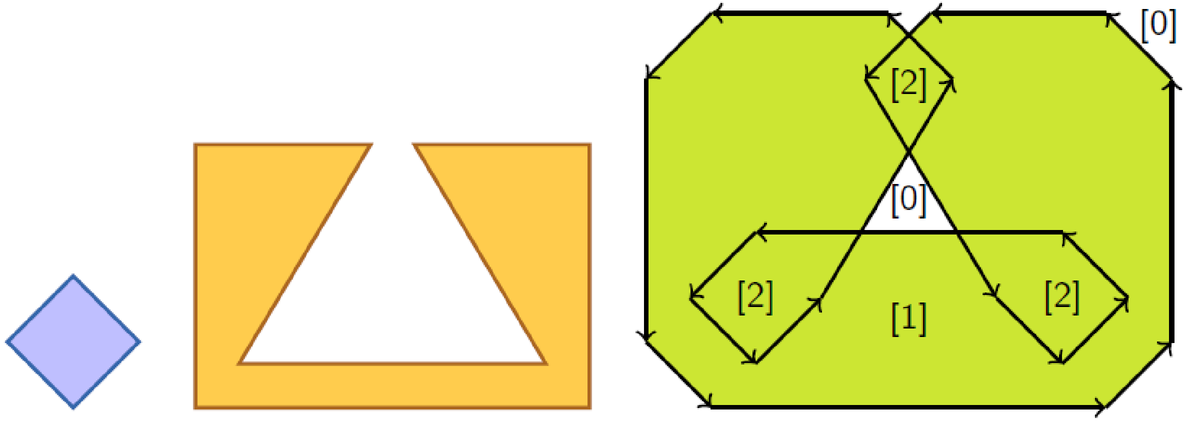
- $P \otimes Q$ — the convolution of P and Q is a collection of line segments:
  - $[p_i + q_j, p_{i+1} + q_j]$, where $\overrightarrow{p_i p_{i+1}}$ lies between $\overrightarrow{q_{j-1} q_j}$ and $\overrightarrow{q_j q_{j+1}}$ and
  - $[p_i + q_j, p_i + q_{j+1}]$, where $\overrightarrow{q_j q_{j+1}}$ lies between $\overrightarrow{p_{i-1} p_i}$ and $\overrightarrow{p_i p_{i+1}}$.

[Guibas-Ramshaw-Stolfi '83]



Construction by convolution

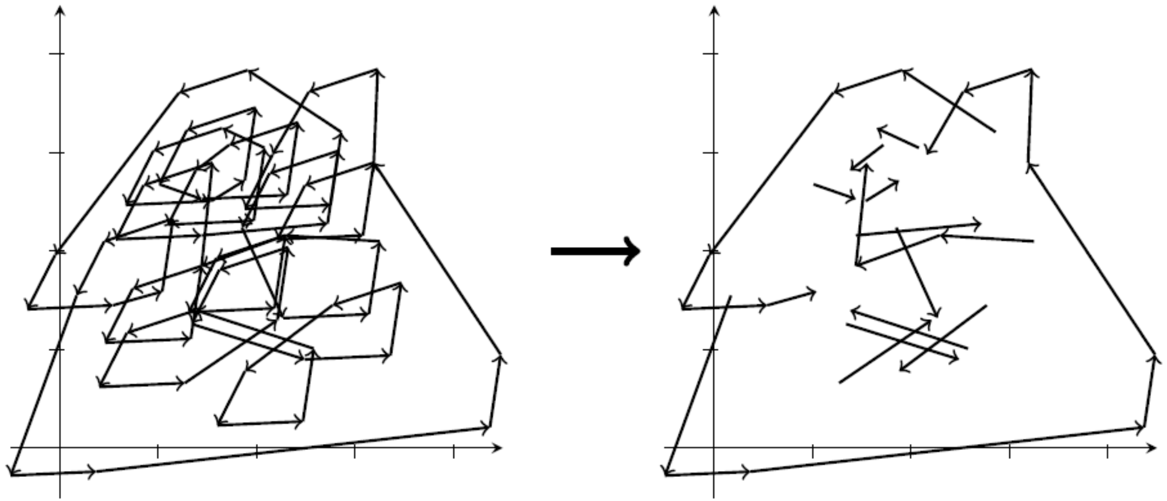
- Lemma: The boundary of the sum is included in the convolution of the boundaries
- Step 1 Track the boundaries simultaneously systematically to create a 2D arrg
- Step 2 Compute winding numbers of faces in the arrg
- Step 3 Construct the union of positive winding-number faces



Speeding up the convolution algorithm in practice by filters

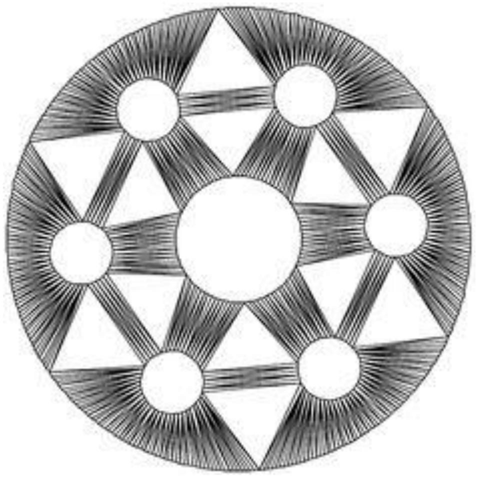
- Construct a partial version of the underlying arrangement
  - Pro: fewer edges → faster arrangement construction
  - Con: we lose the winding number property
- Apply filters; for example, the reflex vertex filter: reflex vertices do not contribute to the Minkowski sum boundary [Kaul-O'Connor-Srinivasan '91]
- Check for each face in the resulting arrangement whether it is inside the sum

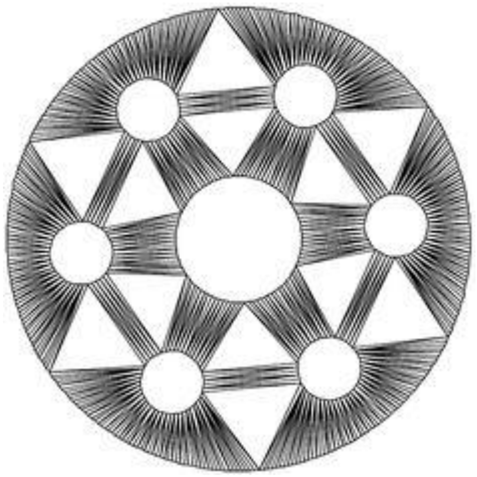
[Behar-Lien '11]



The hole filter

Q: Given two polygons-w/h, which holes can you fill up and still get the same Minkowski sum?

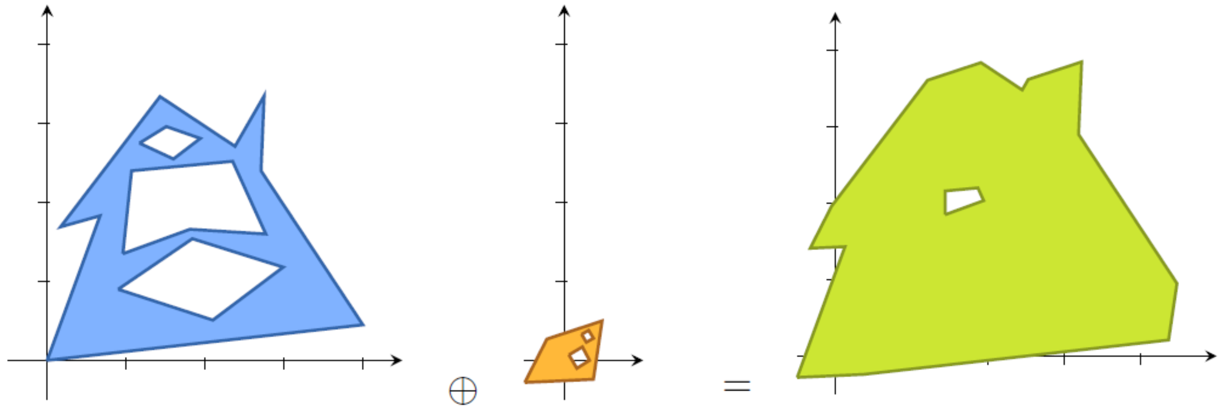




[poly2tri]

The hole filter

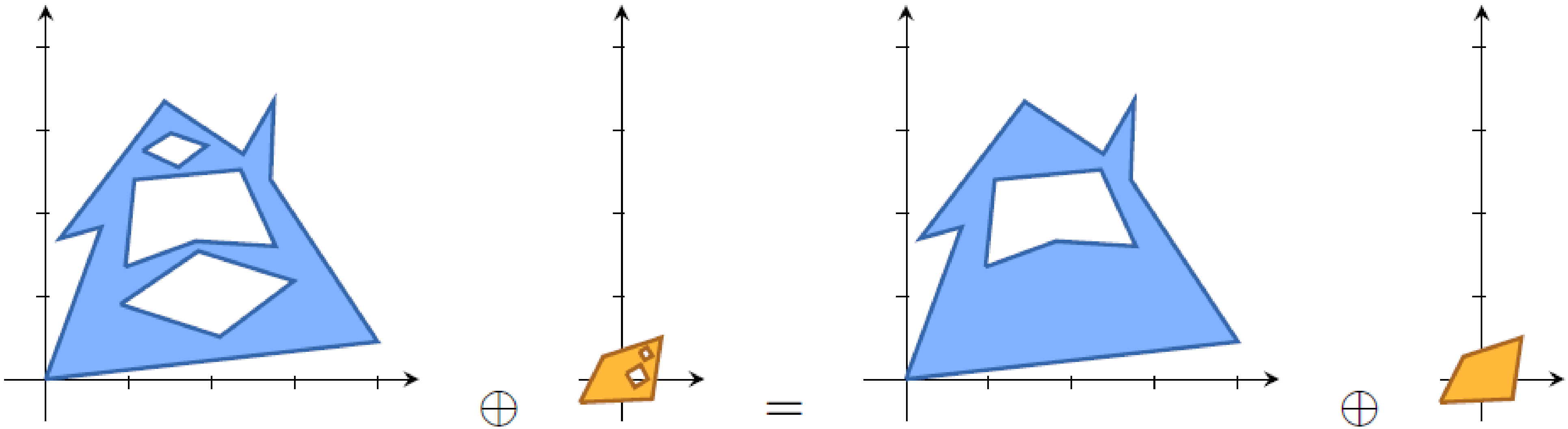
Q: Given two polygons-w/h, which holes can you fill up and still get the same Minkowski sum?



The hole filter, cont'd

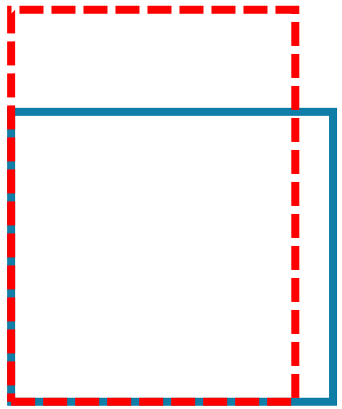
Theorem: Let H be a hole in P. Then

$P \oplus Q \neq (P \cup H) \oplus Q$ iff $\exists t \in R^2$ s.t. $Q \oplus \{t\} \subseteq -H$.



Easily computable filters

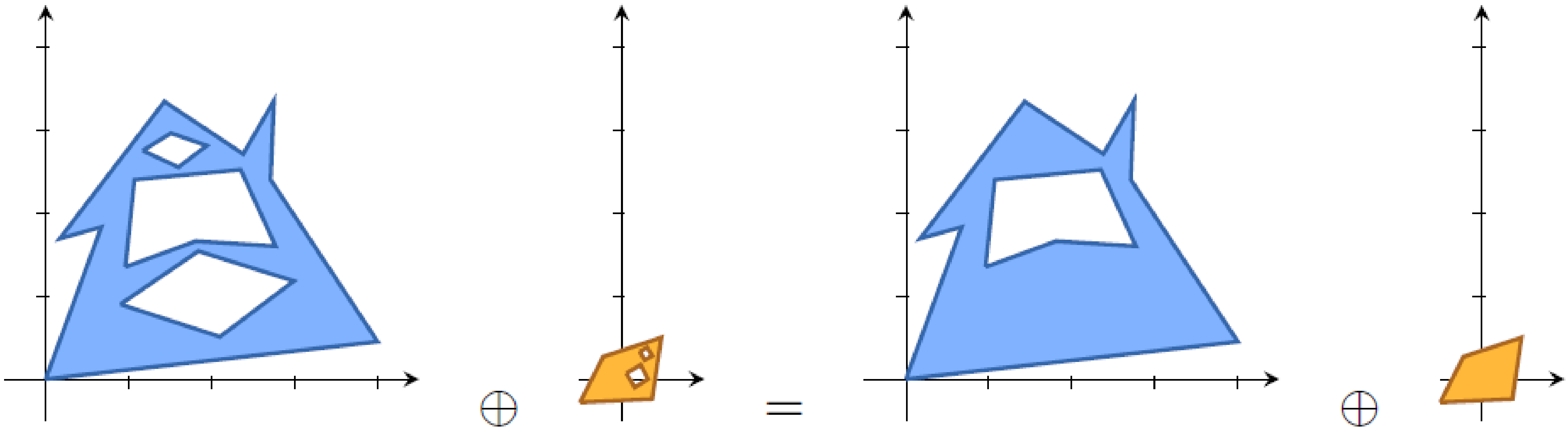
- Check if the hole H in P should be filled up by comparing the axis-aligned bounding box of H and the axis-aligned bounding box of Q



The hole filter, cont'd

Corollary:

One can fill up all the holes of at least one polygon and still get the same Minkowski sum



Hole theorem, proof

Reminder:

Let H be a hole in P. Then

$P \oplus Q \neq (P \cup H) \oplus Q$ iff $\exists t \in R^2$ s.t. $Q \oplus \{t\} \subseteq -H$.

- Recall that $A \cap (B \oplus \{t\}) \neq \emptyset$ iff $t \in A \oplus -B$
- $(\exists t \in R^2 \ldots)$ then

$-Q \oplus \{t\} \subseteq H$

$P \cap (-Q \oplus \{t\}) = \emptyset$, $(P \cup H) \cap (-Q \oplus \{t\}) \neq \emptyset$

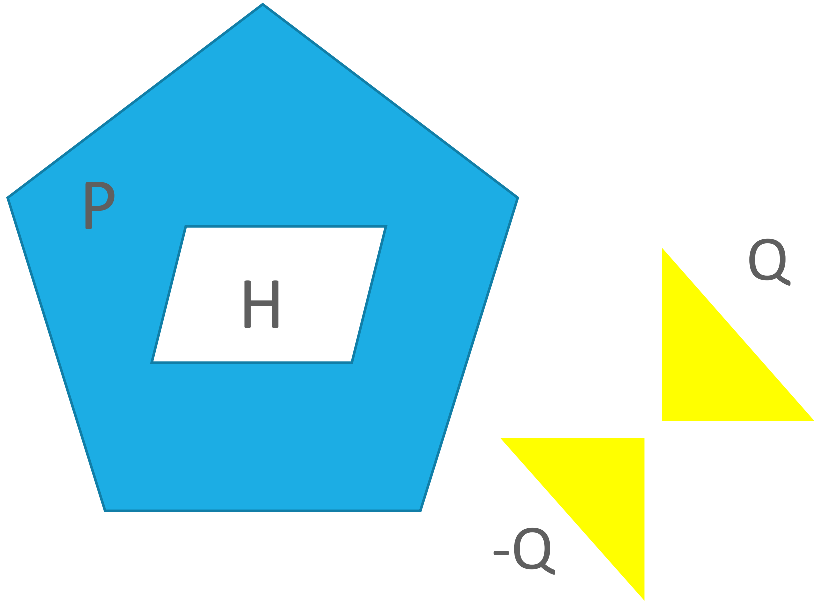
$t \notin P \oplus Q$, $t \in (P \cup H) \oplus Q$

- $(\nexists t \in R^2 \ldots)$ then

$\forall t$, if $(-Q \oplus \{t\}) \cap H \neq \emptyset$ then $(-Q \oplus \{t\}) \cap \partial H \neq \emptyset$, namely $(-Q \oplus \{t\}) \cap P \neq \emptyset$

$t \in (P \cup H) \oplus Q \Rightarrow t \in P \oplus Q$

□



Convex - general

- We saw that when the robot is a convex polygon, the complexity of the free space (complement of the C-obstacles) is favorable: how about algorithms?
- Standard approach: divide-and-conquer, where the merge step uses sweep line to compute the union of two subsets of expanded obstacles
- More efficient approach using medial axis?

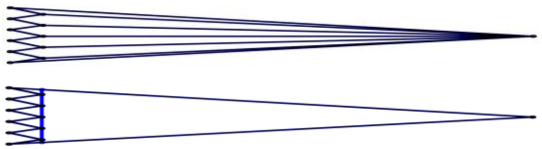
More on the decomposition approach

- Variations based on
  - Convex decomposition
  - Union algorithm

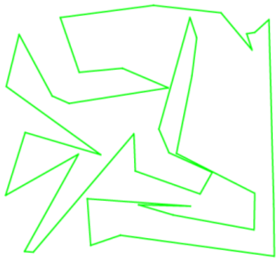
Speeding up the decomposition algorithm

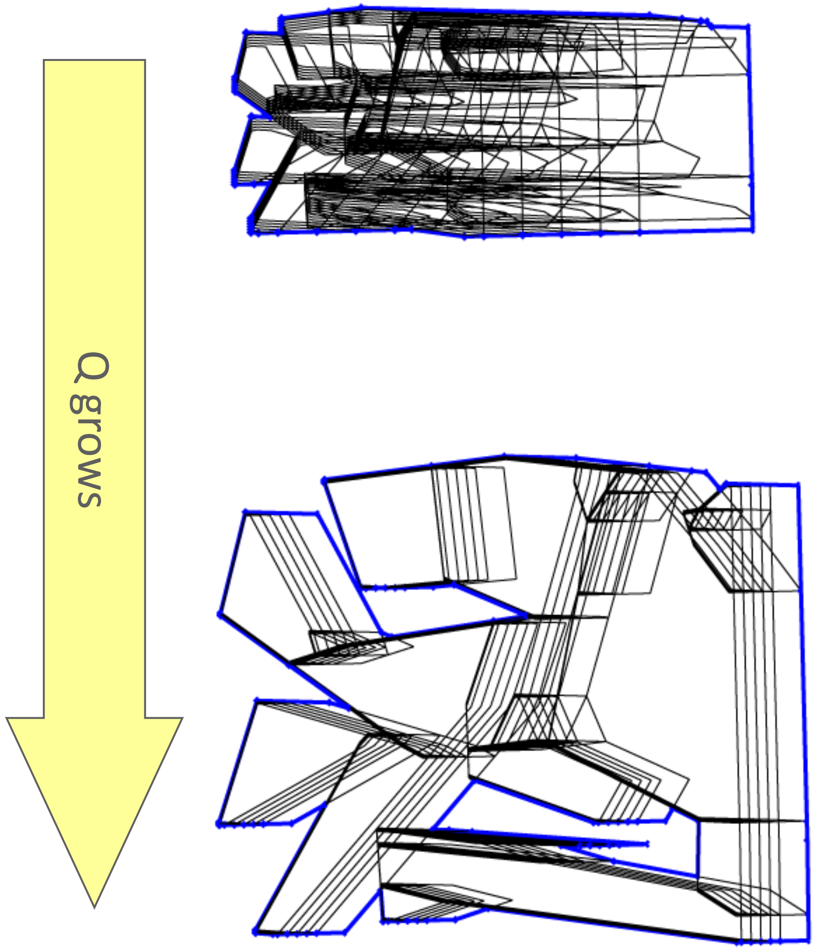
Decomposition length effect: an example

P - fixed size, two types of decompositions

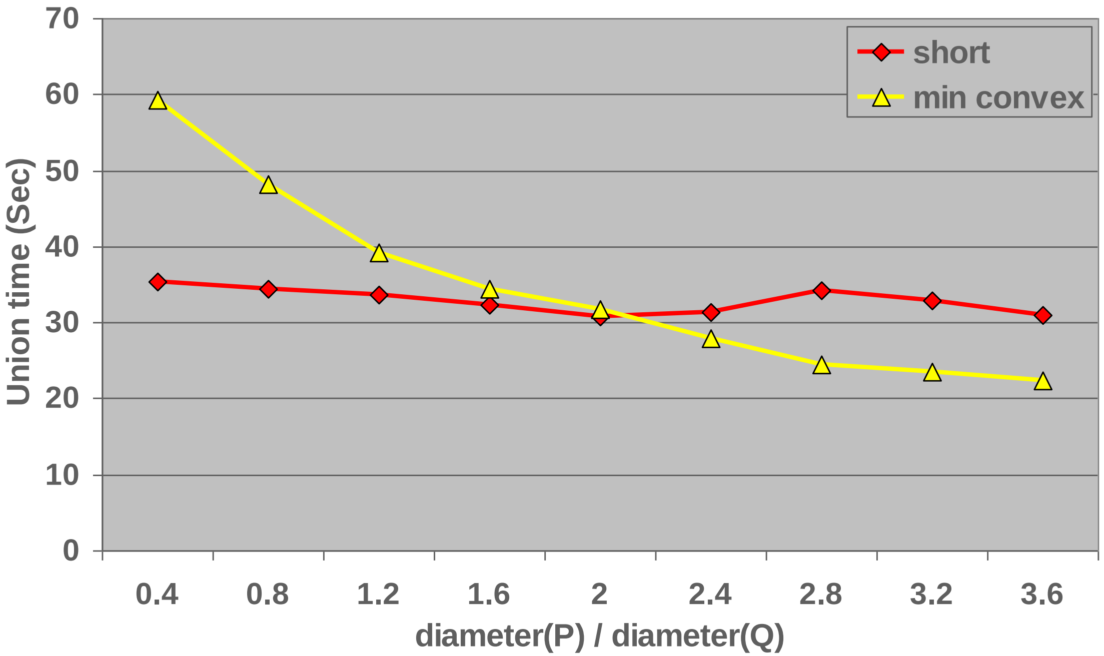


Q - fixed decomposition, scaled size





Decomposition length effect: results

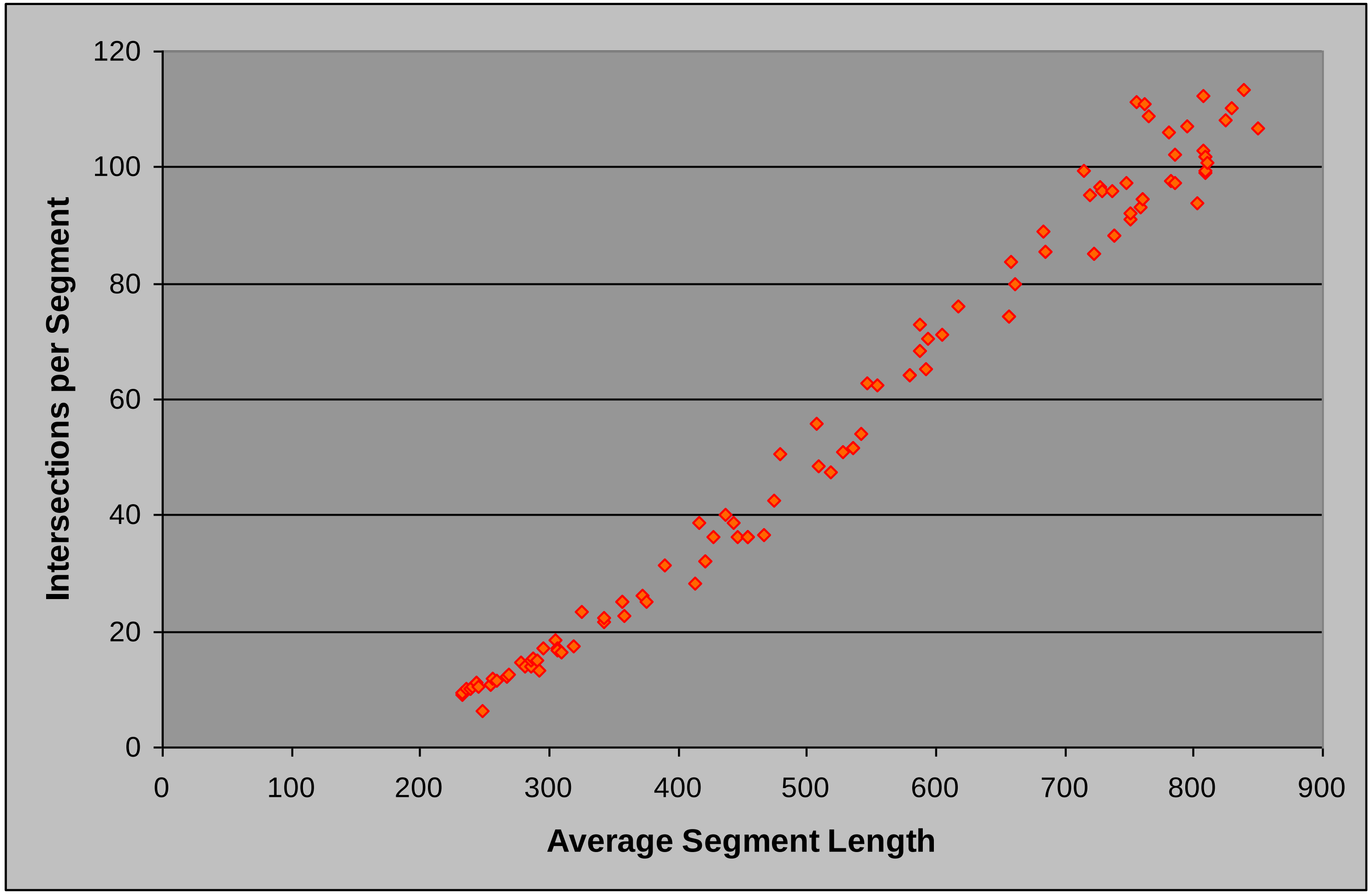


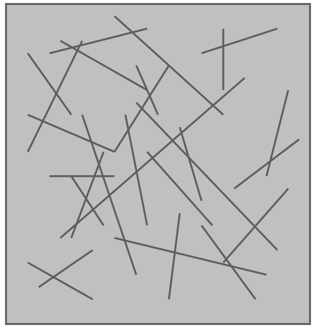
time for computing the Minkowski sum of a knife polygon P (using two types of decompositions) with a random polygon Q that is scaled differently

Smaller number of intersections of segments

- We want each edge of R to intersect as few polygons of R as possible
- $\mu(L(R_{ij}))$ - the standard rigid-motion invariant measure of the set of lines intersecting R_{ij}
- $\mu(L(R_{ij}))$ is the perimeter of R_{ij}

Length vs. number of intersections





Optimizing the mixed objective function

$$k_Q(2\Delta_P + \Pi_P) + k_P(2\Delta_Q + \Pi_Q)$$

k_P - number of subpolygons in the convex decomposition of P

Δ_P - total length of diagonals in the decomposition of P

Π_P - the perimeter of P

The function measures the overall length of the edges of R

An $O(n^2 {r_P}^4 + m^2 {r_Q}^4)$-time decomposition algorithm that minimizes this function

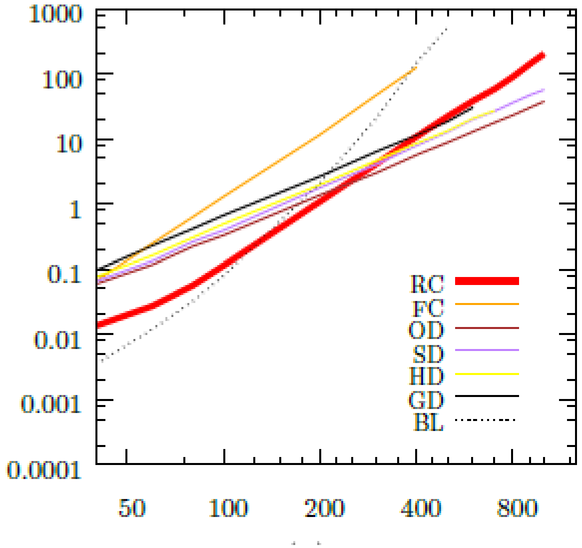
r_P - number of reflex vertices in P

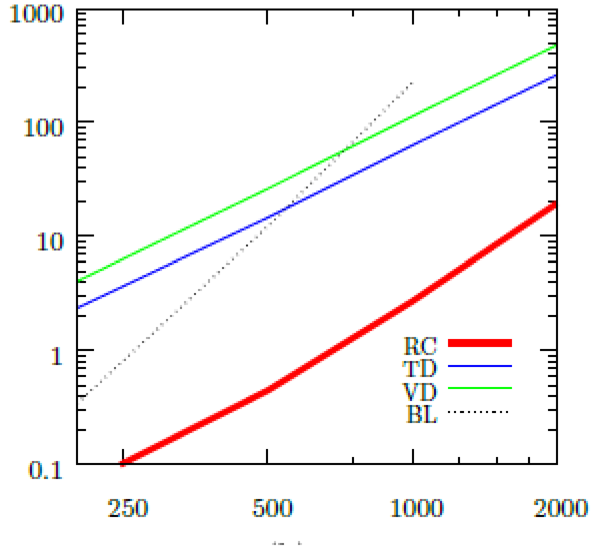
[Agarwal-Flato-H '02]

Implementation

The CGAL Minkowski_sum_2 Package

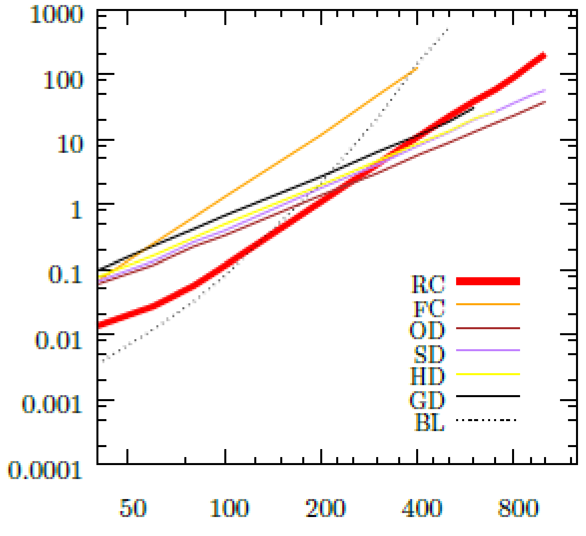
- Based on the Arrangement_2, Polygon_2, and Partition_2 packages
- Our reduced convolution (RC) includes the recent filters: Holes I (the hole theorem) and Holes II (each hole of the sum is part of the convolution of one boundary component from each summand)



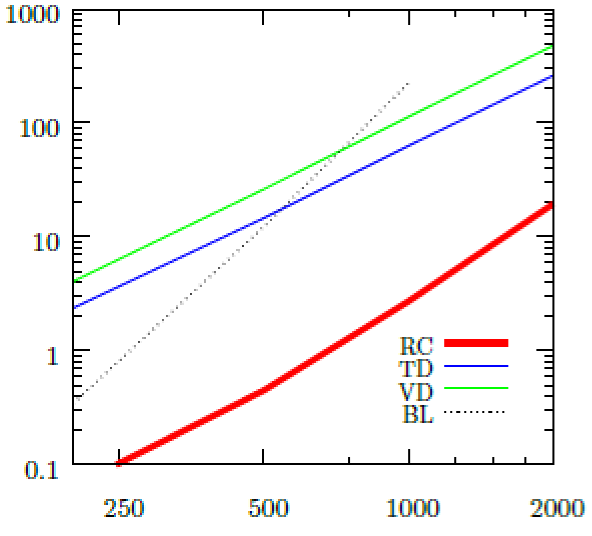


Legend in the next slide

Simple polygons



Polygons with holes



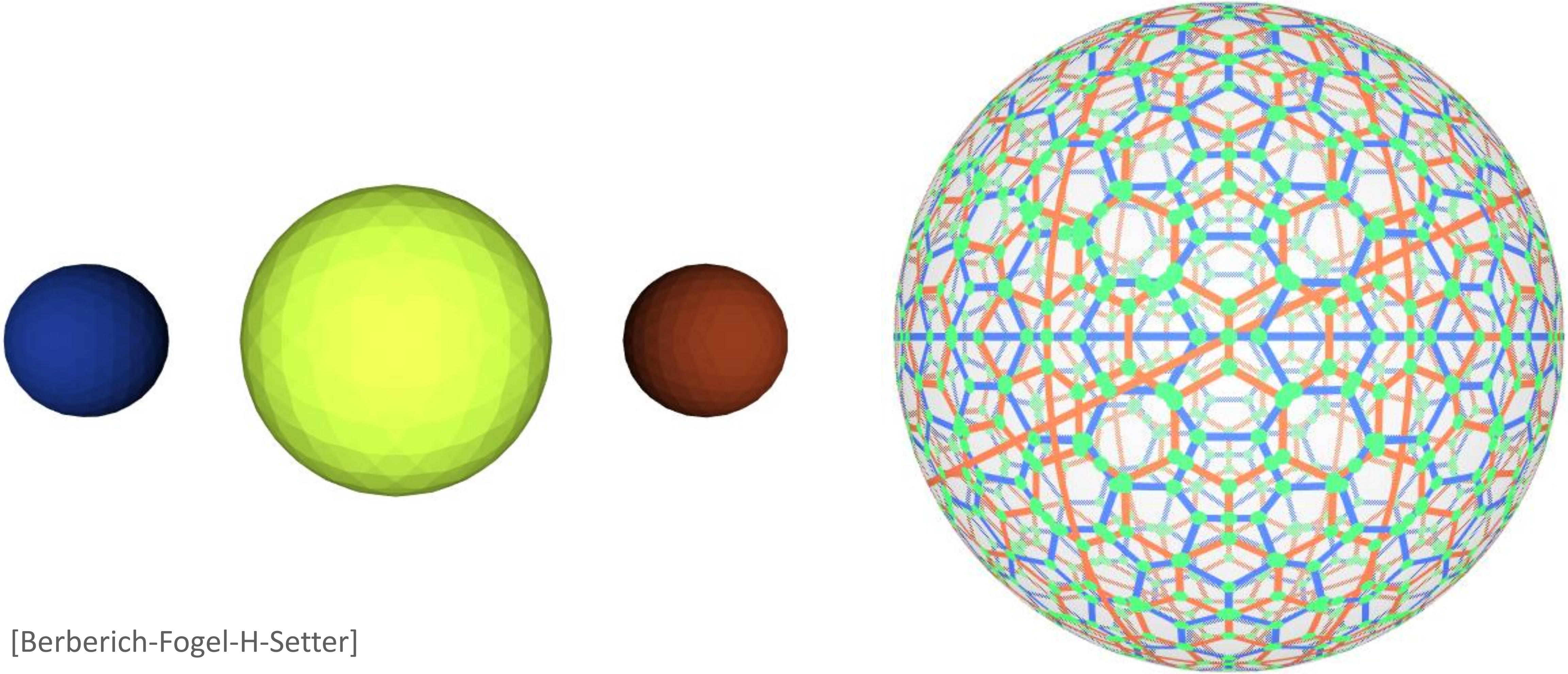
RC: our reduced convolution
TD: our triangular decomposition
VD: our vertical decomposition
BL: Behar and Lien's RC

- The consumption time in seconds as a function of the # of vertices
- MS of pairs of polygons with holes with n vertices and n/10 holes
- More results in the paper

[Baram-Fogel-H-Hemmer-Morr]

3D

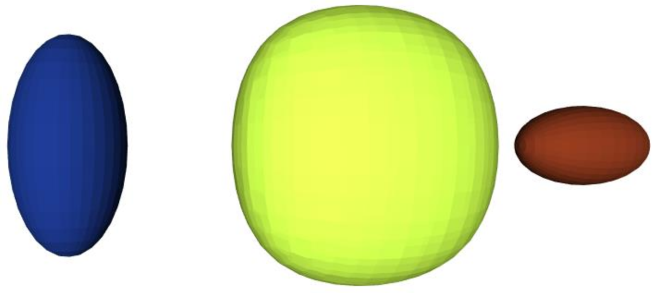
Convex polytopes and spherical arrangements



[Berberich-Fogel-H-Setter]

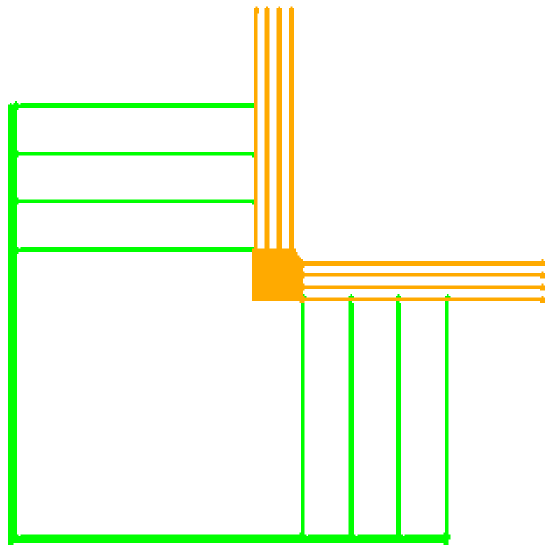
Convex polytopes

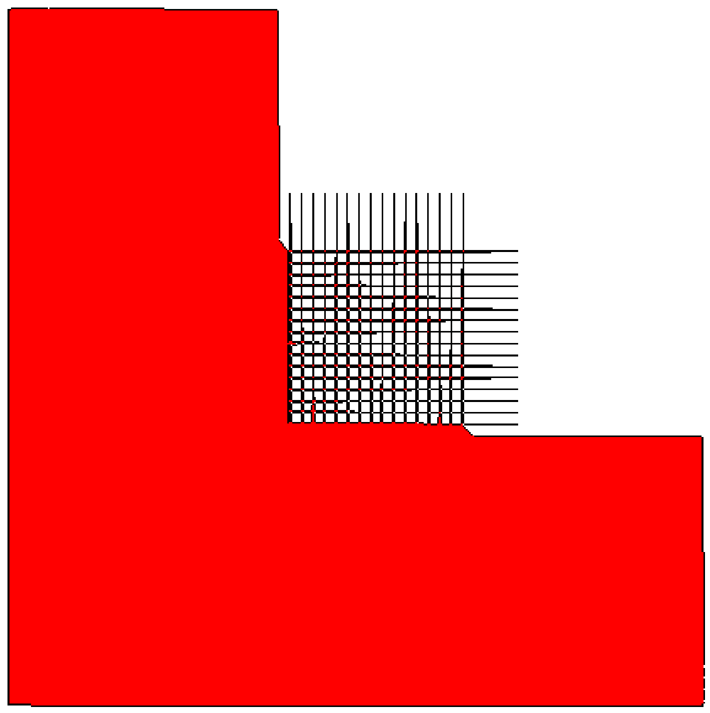
- Recall that for polygons with m and n vertices, the sum has at most $m + n$ vertices
- For polytopes (3D) with m and n vertices, the sum has $\Theta(mn)$ vertices; exact numbers [Fogel-H-Weibel '09]



Arbitrary polyhedra

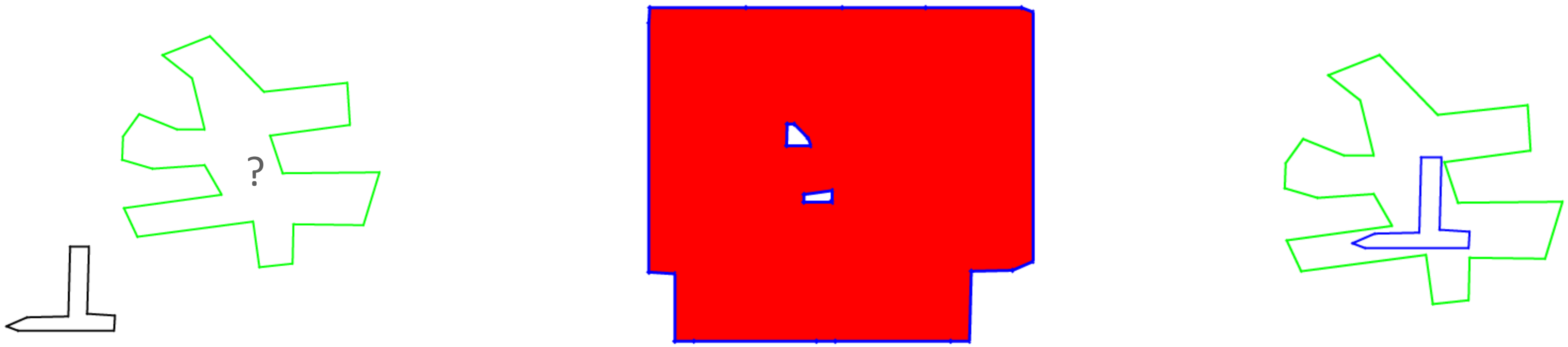
- Recall that for polygons with m and n vertices: $O(m^2n^2)$
- Polyhedra with m and n vertices: $O(m^3n^3)$
- These bounds are tight



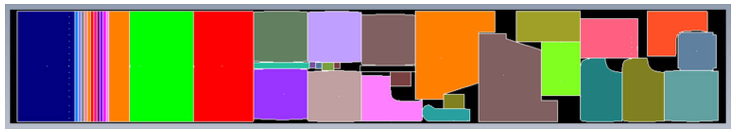


Minkowski sums, more applications

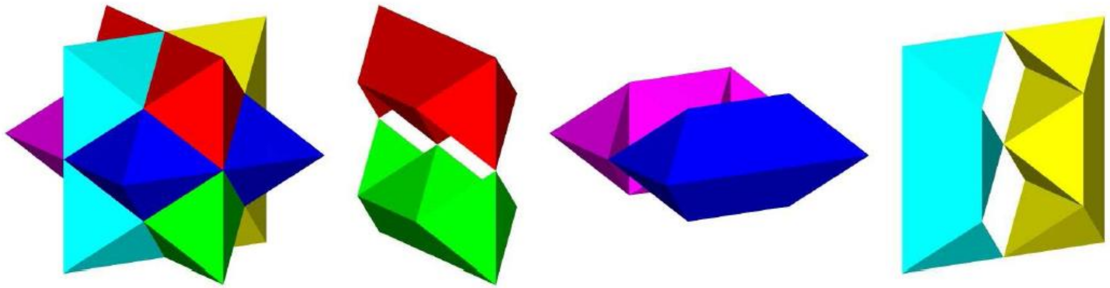
- Minimum separation distance (penetration depth)
- Placement



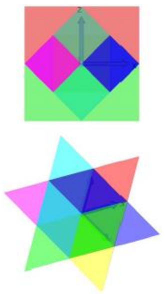
- Tolerancing, offsetting
- Nesting
- Cartographic generalization

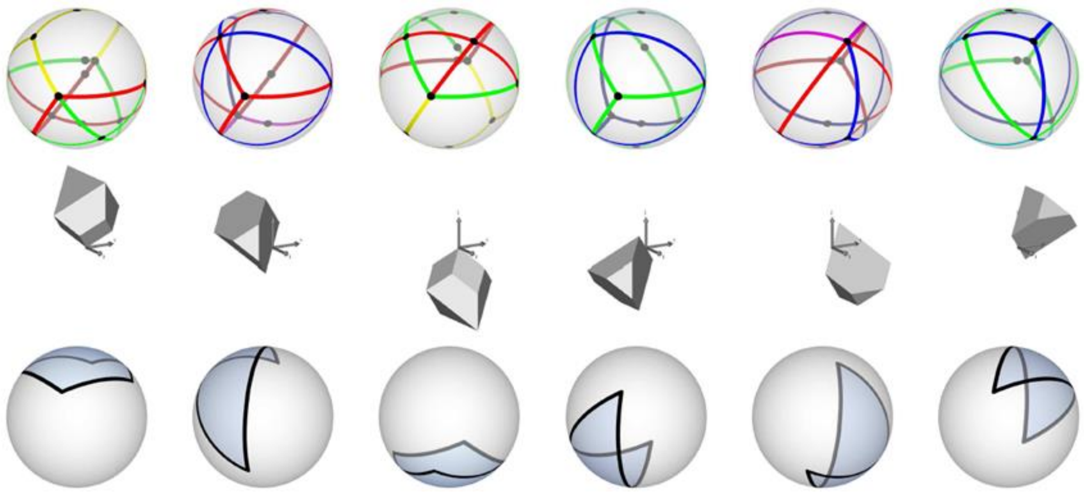


Assembly planning



the splitStar puzzle





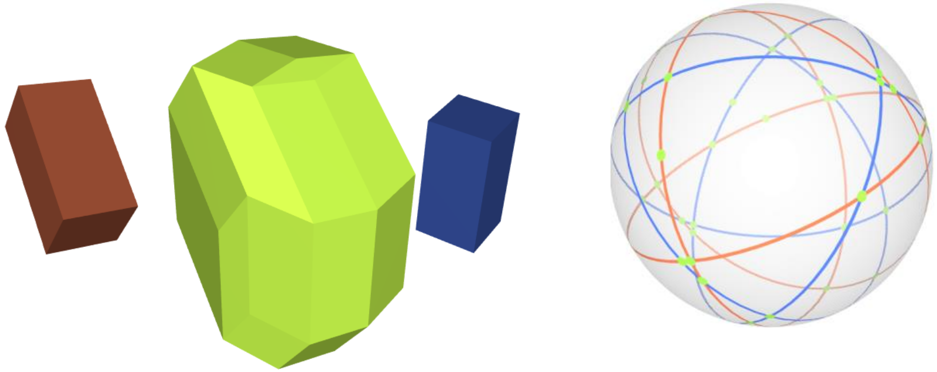
projection of Minkowski sums onto the sphere

[Fogel-H ‘13]

Standard operations

- Minimum separation distance (penetration depth)

- The Separating Axis Theorem: Given two boxes in 3D (OBBs), one can decide if they collide by testing their projection along 15 lines [Gottschalk et al. '96]
- Which are these 15 lines and why are 15 sufficient?
  - Consider the Minkowski sum of the two OBBs



Minkowski average, riddle

Let A be a regular polyhedral set in R^d.

Consider the sequence $A, \frac{A \oplus A}{2}, \frac{A \oplus A \oplus A}{3}, \ldots$

What can we say about $\frac{A \oplus A \oplus \ldots \oplus A}{k}$, where A appears k times in the numerator, as k goes to infinity?

Minkowski average, solution

[Shapley-Folkmann-Starr ‘69]

Minkowski average, convergence

Consider the sequence $A, \frac{A \oplus A}{2}, \frac{A \oplus A \oplus A}{3}, \ldots$ in R^d.

Does the volume monotonically increase?

For $d = 1$, yes

For $d \geq 12$, no

For $1 < d < 12$, ?

[Fradelizi et al, ‘16]

Open problems and challenges

The mystery of the construction time

Challenges

- Quasi output-sensitive algorithms
- More filters: Given P and Q, what is the family of P's such that $P' \oplus Q = P \oplus Q$

Major engineering challenge: Exact and efficient implementation of the general 3D case

- The decomposition method [Hachenberger ‘09]
- 3D arrangements
  - Collins decomposition
  - Decomposition-sensitive sweep [H-Shaul ‘02]
  - Central difficulty: Degeneracies
  - Geometric Rounding

- Alternative approaches: Lien et al, Manocha et al

Reference for the convex – non-convex case

- Computational Geometry: Algorithms and Applications, de Berg et al, 3rd Edition, Springer, 2008

  Chapter 13: Robot motion planning

THE END