Algorithmic Robotics and Motion Planning

Piano Movers I, Schwartz-Sharir '83
From Latombe's book

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Translation and rotation in the plane

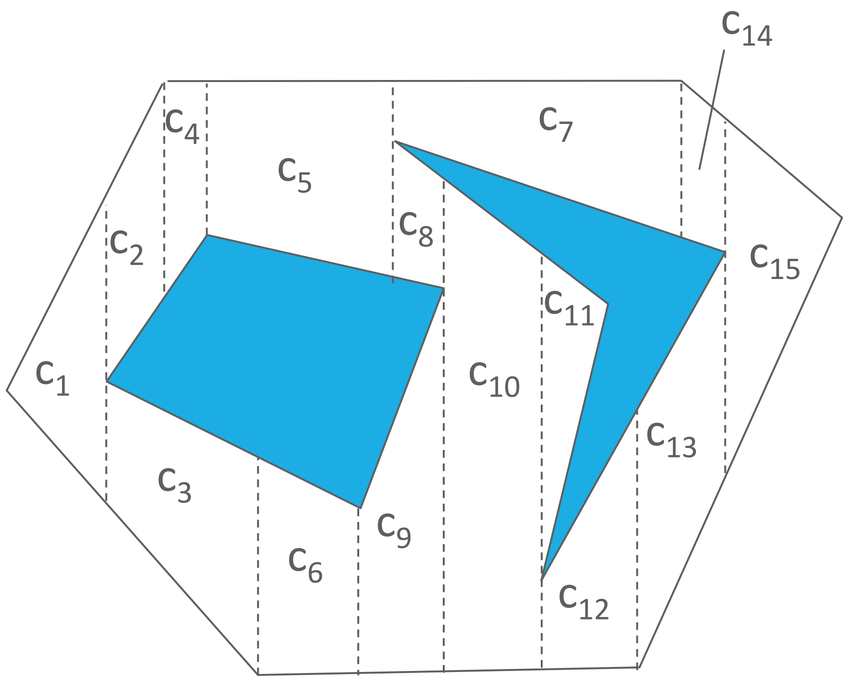
- C-space is three dimensional
- Boundaries of C-obstacles are no longer linear

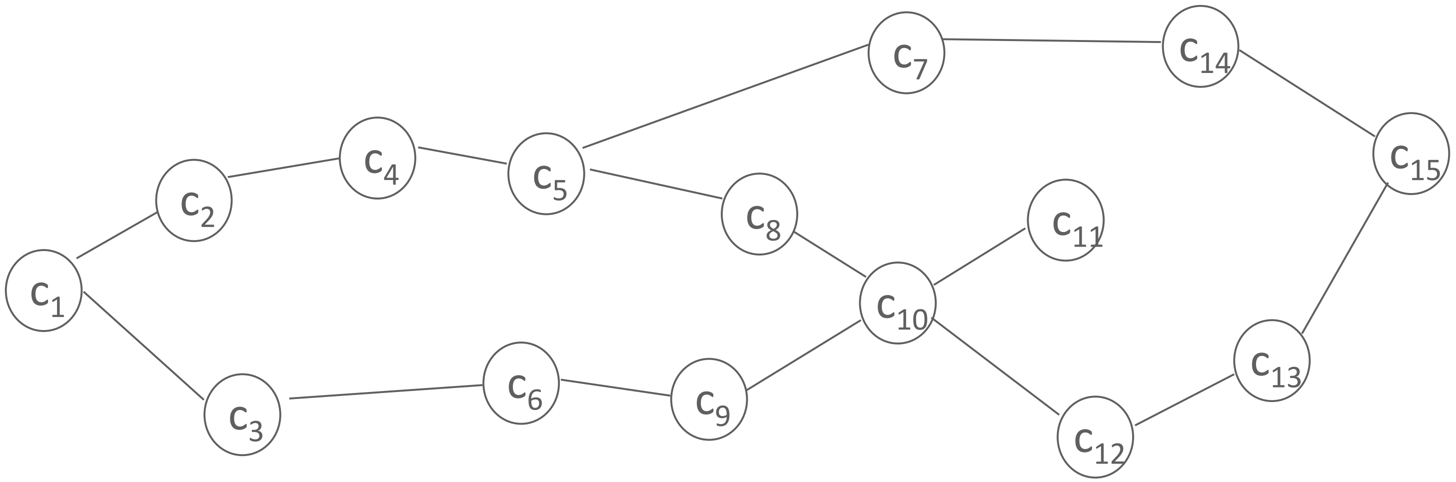
Today’s lesson

- Preliminaries
- Piano Movers I: The case of a ladder
- The implications of Piano Movers I

Preliminaries

Reminder: The connectivity graph



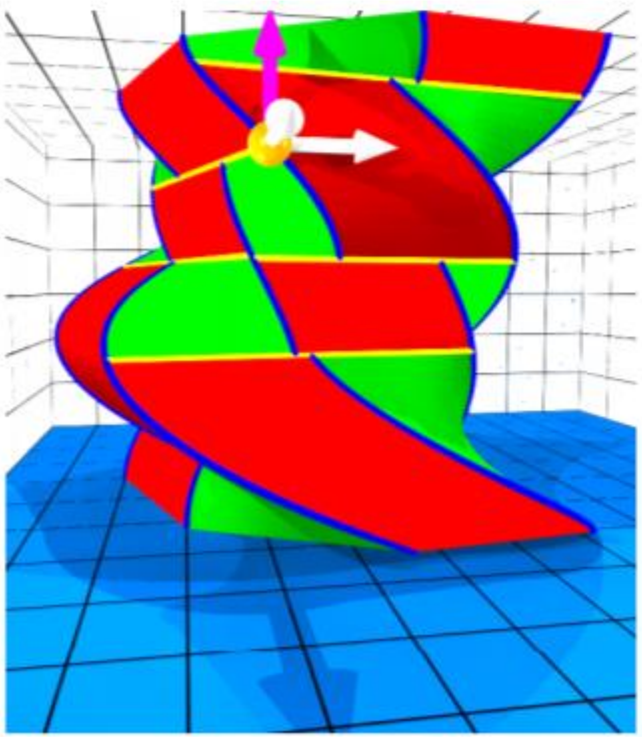


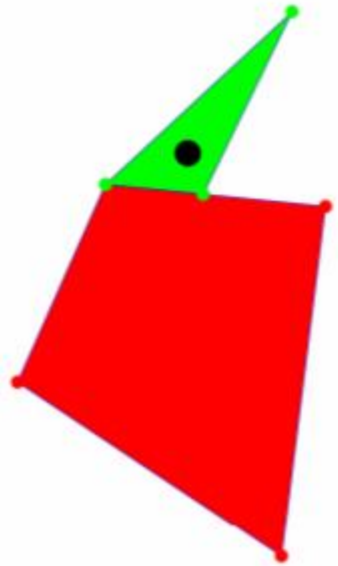
[www.seas.upenn.edu/~jwk/motionPlanning]

Arrangements of critical surfaces, Take I

- arrangements $\mathcal{A}(\mathcal{S})$ are used for exact discretization of continuous problems
- a point p in configuration space $\mathcal{C}$ has a property $\Pi(p)$
- if a neighborhood U of p is not intersected by an object in $\mathcal{S}$, the same property $\Pi(q)$ holds for every point $q \in U$
(the same holds when we restrict the configuration space to an object in $\mathcal{S}$)
- the objects in $\mathcal{S}$ are critical
- the property is invariant in each cell of the arrangement
- examples

The shape of C-obstacles for a polygon translating and rotating among polygons in the plane





[Atariah-Rote 2012]

[Configuration space visualization]

PIANO MOVERS I:
TRANSLATING AND ROTATING A LADDER AMONG POLYGONAL OBSTACLES

The problem

- The robot A is a line segment PQ, rotating and translating in the plane
- Moving among polygonal obstacles B with a total of n vertices
- C-space: using P as a reference point

- We assume *general position*

Critical curves in the xy-plane

- The projection of the following curves onto the xy-plane
  - The boundaries of faces of C-obstacles
  - Curves of C-obstacles of vertical tangency: curves on faces of the C-obstacles where the tangent plane to the C-obstacles is perpendicular to the xy-plane

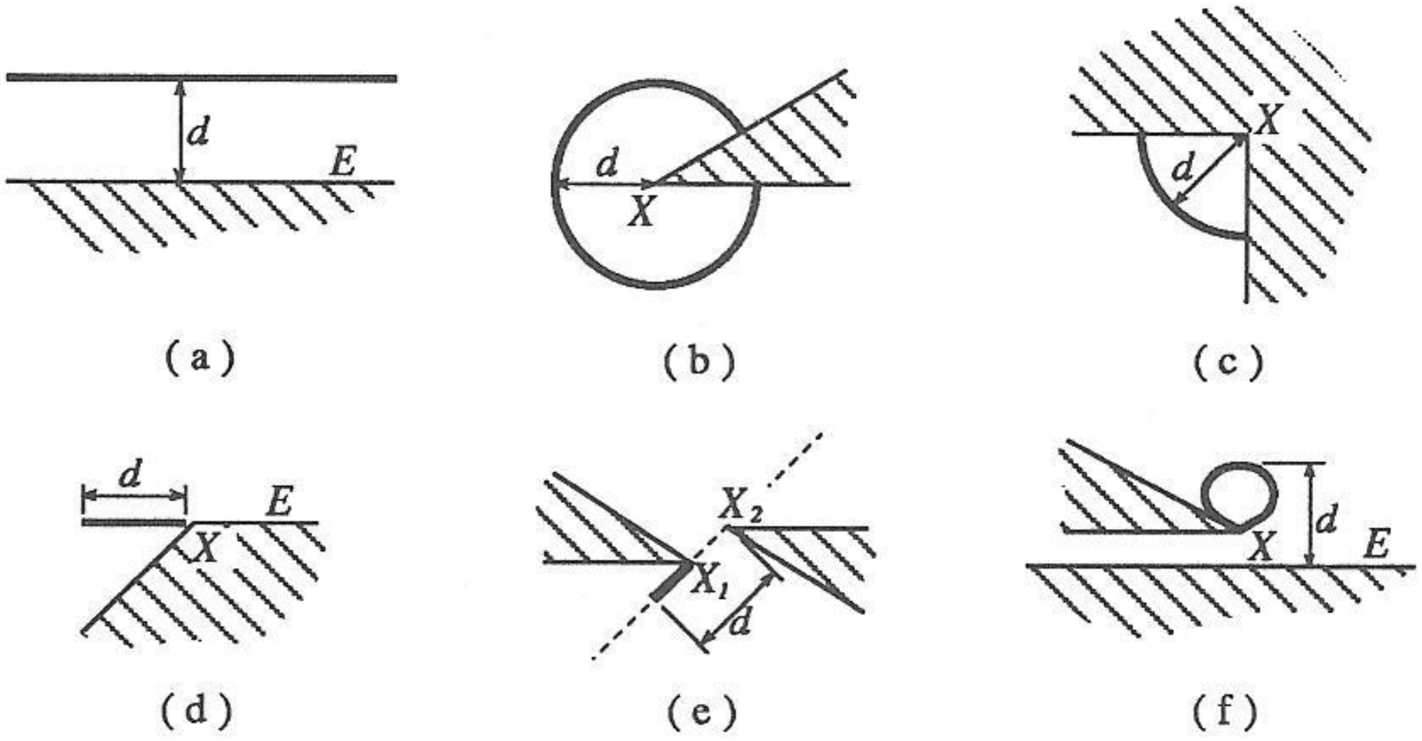


Figure 5. This figure illustrates the various types of critical curves other than the obstacle edges. The critical curves (shown in bold lines) are the set of positions of $\mathcal{A}$ where the structure of the C-obstacle region along the θ direction undergoes a qualitative change.

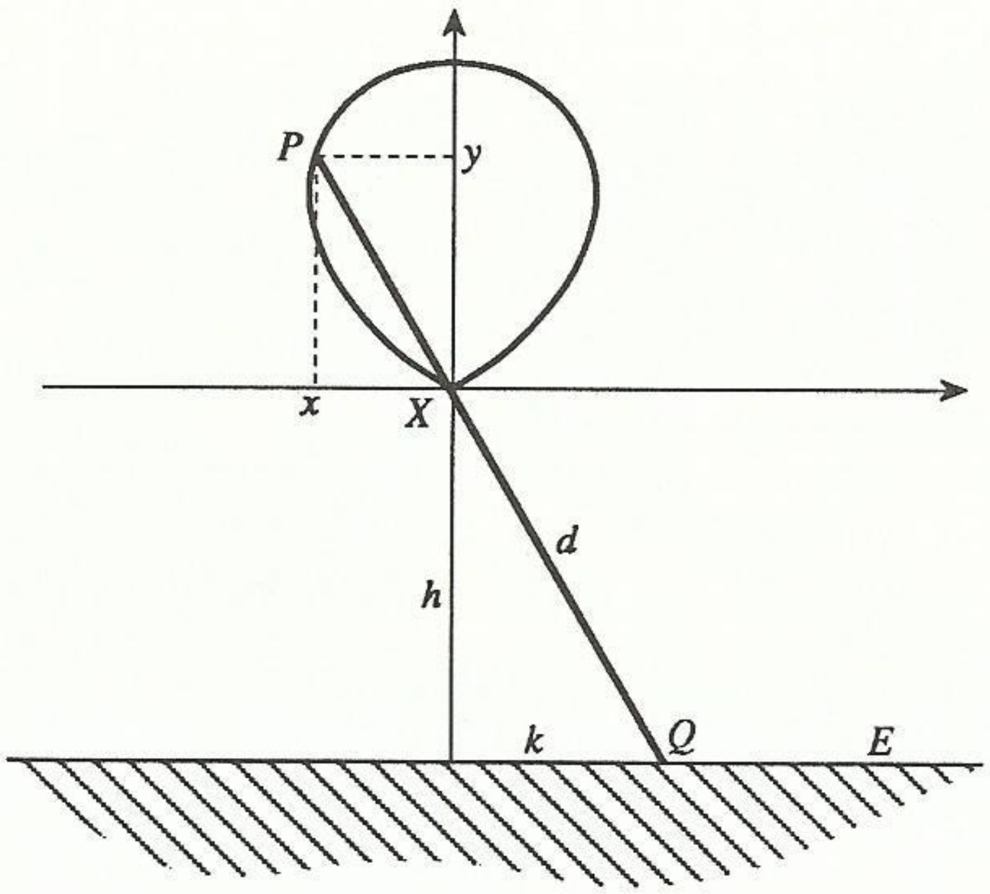


Figure 6. A conchoid of Nicomedes (see text) is an algebraic curve of degree 4. We have: $d^2 = (y+h)^2 + (x+k)^2$ and $\frac{y}{x} = \frac{h}{k}$. Thus: $x^2 = y^2(\frac{d^2}{(y+h)^2} - 1)$.

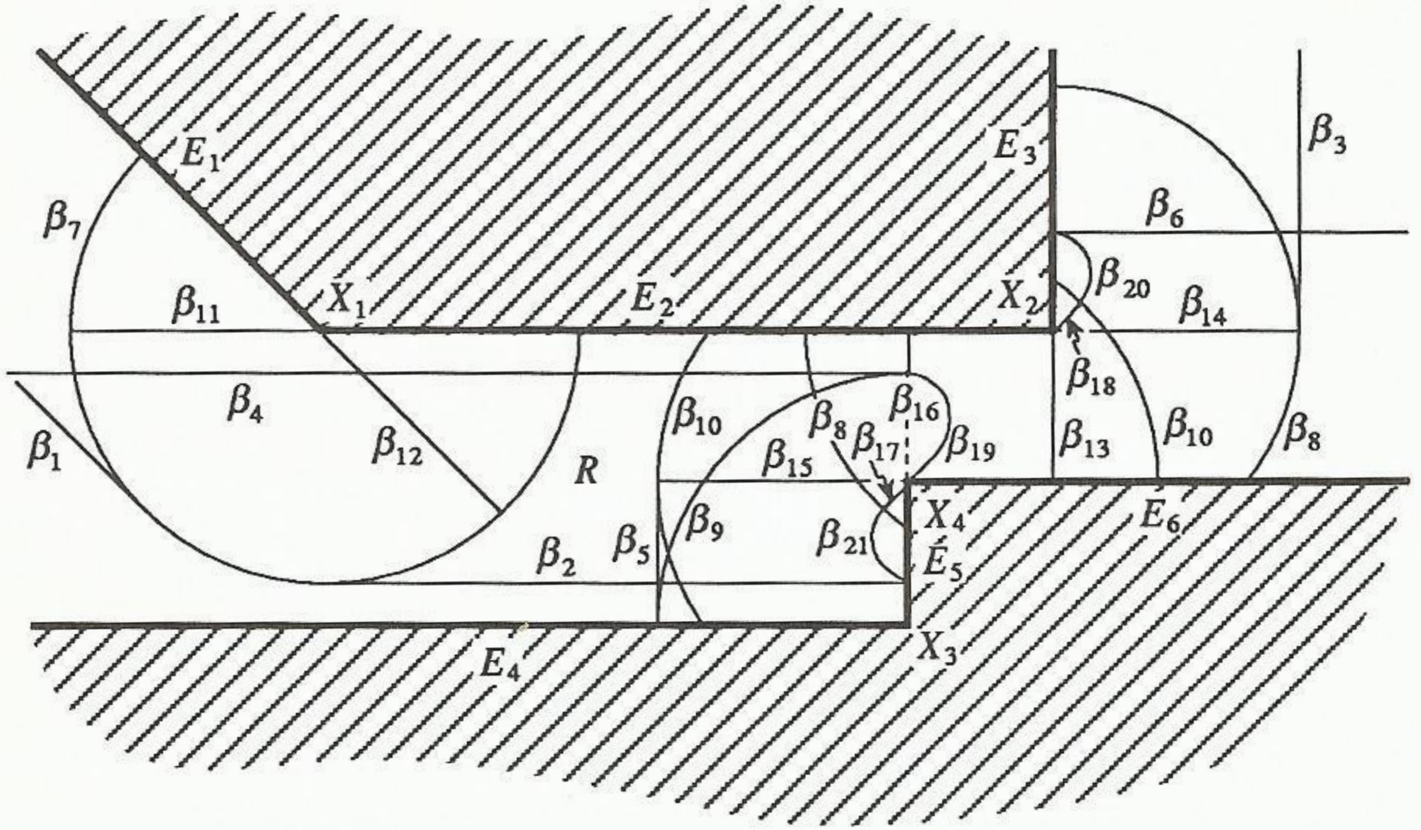


Figure 7. This figure illustrates the concepts of a critical curve and a non-critical region in a polygonal workspace. The boundaries of the obstacles are shown in bold lines. The critical curves of types 1 through 5 are shown in plain thin lines. A subset of the critical curve β_{16} is shown in dashed lines and corresponds to a "redundant" critical curve section. Any open region enclosed by critical curves, e.g. R, is a noncritical region.

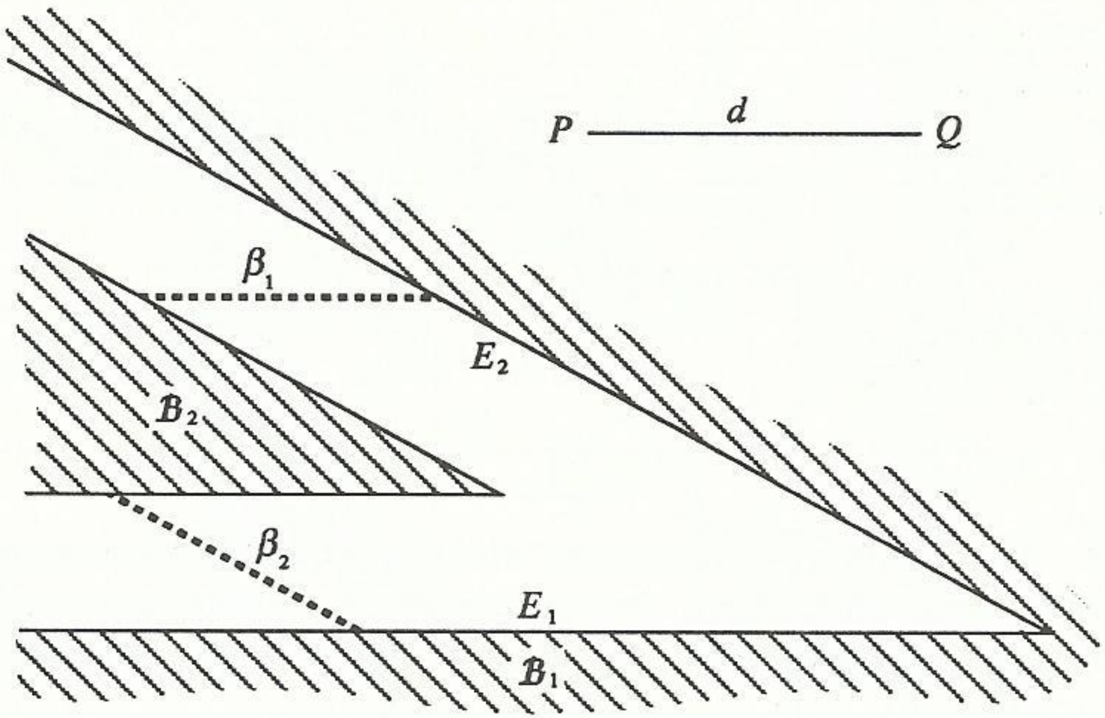


Figure 8. This figure illustrates the notion of a redundant critical curve section in a subset of $\mathcal{W}$ that contains two obstacles $\mathcal{B}_1$ and $\mathcal{B}_2$ forming a "corner". Two redundant critical curves of type 1, β_1 and β_2, are shown in dashed lines. β_1 (resp. β_2) is the locus of P when Q is in E_1 (resp. E_2) and PQ is perpendicular to E_1 (resp. E_2). However, in both cases, PQ intersects with $\mathcal{B}_2$, implying that β_1 and β_2 are the projections of curves contained in the C-obstacle $\mathcal{CB}_2$.

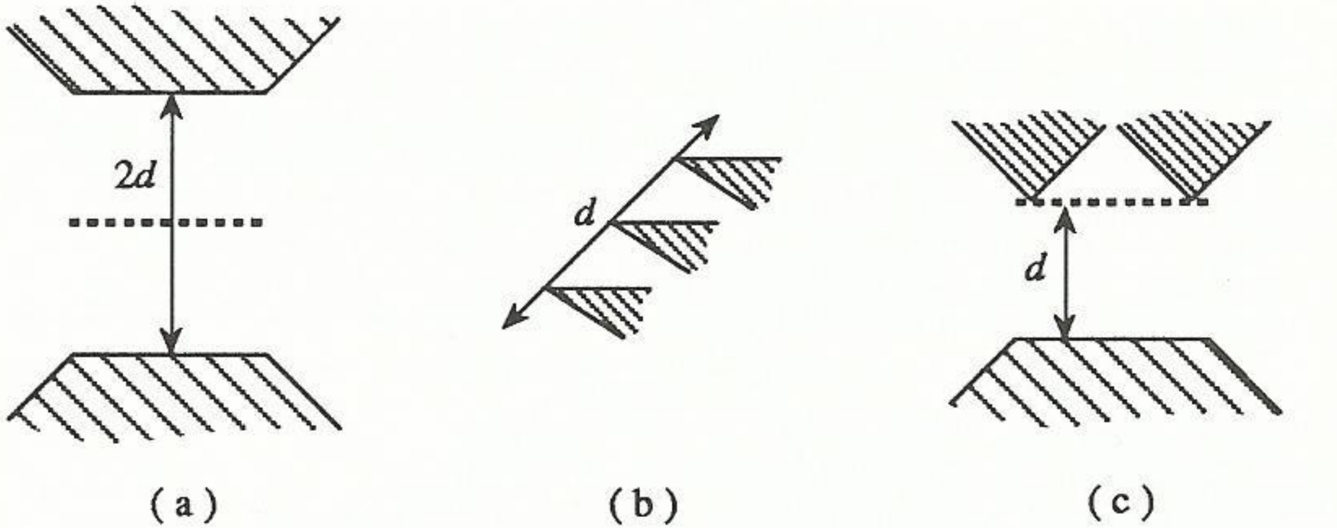


Figure 9. This figure illustrates cases where several critical curves coincide over a section of non-zero length. In our presentation we assume that there is no such coincidence. In theory, a coincidence can be eliminated by changing the position of an obstacle by an arbitrarily small amount.

Noncritical region

- A position (x,y) is admissible if there is at least on orientation θ such that (x,y,θ) is in C_{free}
- A noncritical region is a maximal subset of admissible positions of the robot, which intersects no critical curve
- An admissible position (x,y) is noncritical iff it lies in a noncritical region, otherwise it is critical

Free orientations

- Let (x,y) be a noncritical position
- F(x,y) is the set of all free orientations at (x,y)
- Either all orientations are free, namely F(x,y)=[0,2π), or
- F(x,y) comprises a finite number of open maximal connected intervals
- Each interval endpoint θ_c is such that A(x,y, θ_c) touches B's boundary
- θ_c is called a limit orientation of A at (x,y)

Stops

- Assume F(x,y)$\neq$ [0,2π)
- For each maximal interval (θ_1, θ_2) in F(x,y) let s_1 be the vertex or open edge of B touched by A(x,y,θ_2) called clockwise stop, and s_2 be the vertex or open edge of B touched by A(x,y,θ_2) called counterclockwise stop
- Let θ_c be a limit orientation at a noncritical position (x,y)
- s(x,y,θ_c) is the unique stop touched by A(x,y,θ_c)
- σ(x,y) is the ordered list of all pairs [s(x,y,θ_c),s(x,y,θ'_c)] of stops of F(x,y)
- If F(x,y)= [0,2π), we let σ(x,y) :={[Ω,Ω]}

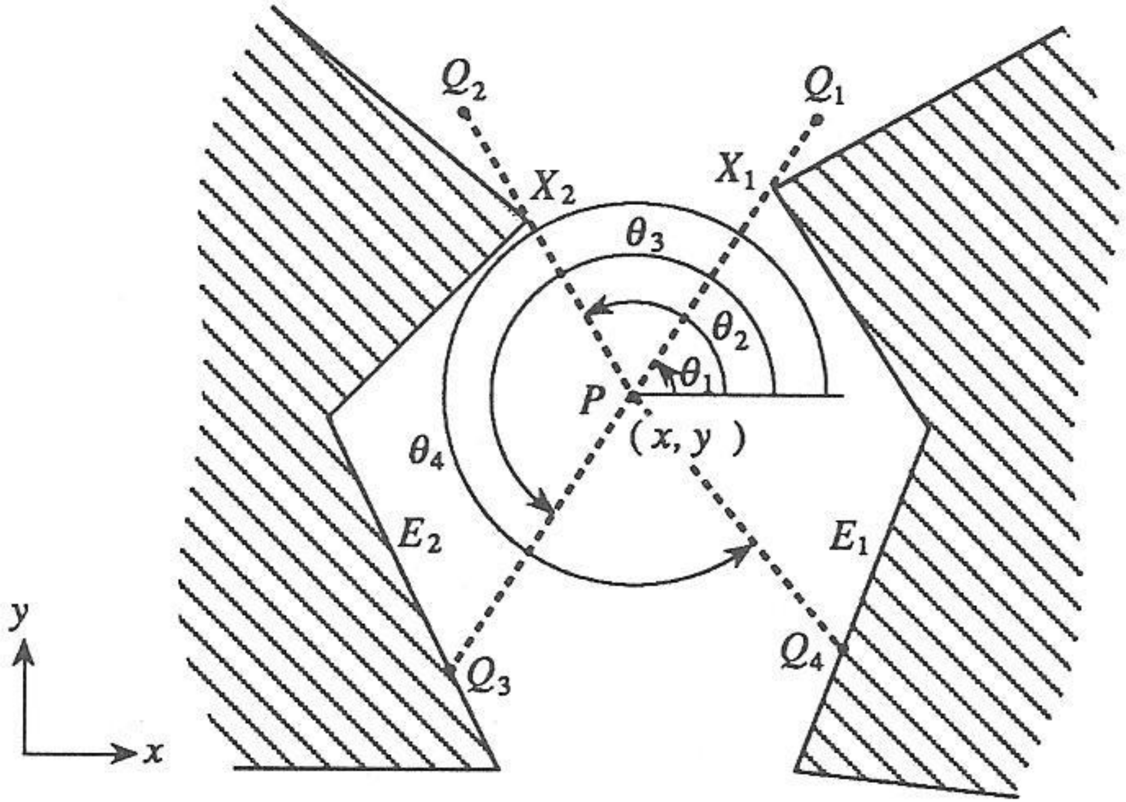


Figure 10. $\mathcal{A}$ is at a noncritical position (x, y). The obstacle edges and vertices it can touch without intersecting the interior of $\mathcal{B}$ when it rotates about P are called stops. A stop is (counter)clockwise if it can be reached from a free configuration by a (counter)clockwise rotation. X_1 and E_2 are clockwise stops, while X_2 and E_1 are counterclockwise stops. The limit orientations corresponding to the stops X_1, X_2, E_2, and E_1 are θ_1, θ_2, θ_3, and θ_4, respectively.

Intervals of free orientations at (x,y)

- Given a pair $[s_1,s_2]$ in $\sigma(x,y)$ we denote by $\lambda_1(x,y,s_1)$ the unique orientation such that $A(x,y, \lambda_1(x,y,s_1))$ touches the clockwise stop s_1
- Similarly, we denote by $\lambda_2(x,y,s_2)$ the unique orientation such that $A(x,y, \lambda_2(x,y,s_2))$ touches the clockwise stop s_2
- If $[s_1,s_2]$ in $\sigma(x,y)$ then $(\lambda_1(x,y,s_1), \lambda_2(x,y,s_2))$ is a maximal connected interval in $F(x,y)$
- Notice that if R is a noncritical region than for every pair of points (x,y), (x',y') in R it holds that $\sigma(x,y)= \sigma(x',y')$
- We therefore let $\sigma(R):= \sigma(x,y)$ for any (x,y) in R

Cells

- R noncritical region, $[s_1,s_2]$ is a pair in σ(R). The 3D region

cell(R,s_1,s_2):=$\{((x,y,\theta) \mid (x,y) \in R$ and $\theta \in (\lambda_1(x,y,s_1), \lambda_2(x,y,s_2))\}$

is called a cell

- *cell*(R,s_1,s_2) is an open connected subset of C_{free}
- For every cell κ=*cell*(R,s_1,s_2) and a point $(x,y) \in R$, we define
  - $\phi_\kappa(x,y)= \lambda_1(x,y,s_1)$, and
  - $\psi_\kappa(x,y)= \lambda_2(x,y,s_2))$
- $\phi_\kappa(x,y)$ and $\psi_\kappa(x,y)$ are continuous fucntions over R

Union of cells

- The set of all cells *cell*(R,s_1,s_2) forms a decomposition of C_{free}
- The cells are disjoint and the closure of their union equals the closure of C_{free}

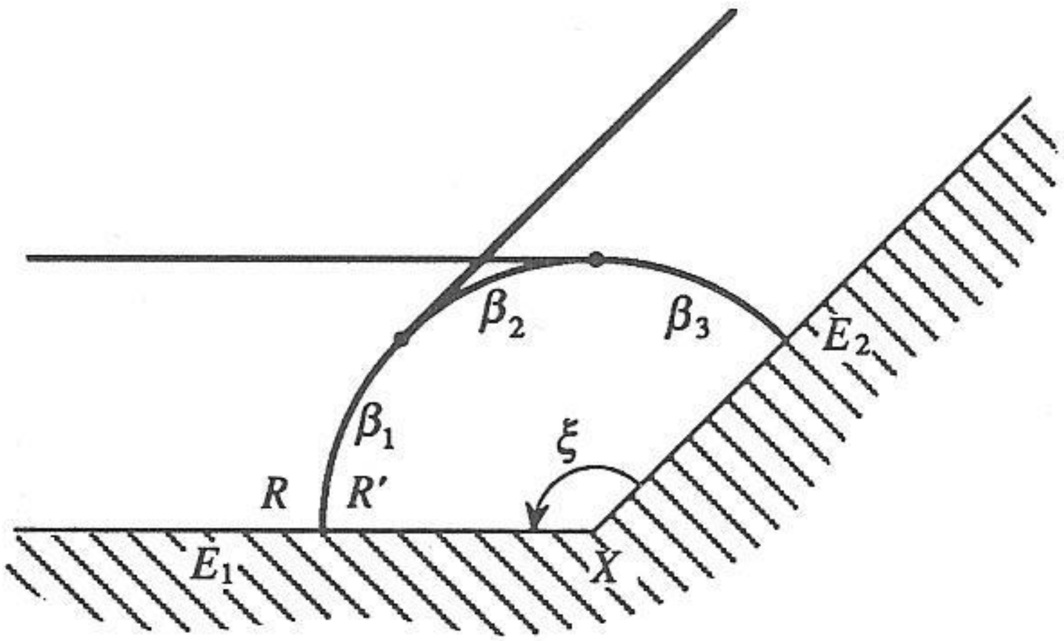


Figure 11. This figure illustrates the notations used to establish the crossing rule of a section of critical curve of type 2.

The connectivity graph

- Each cell is a node
- Connect *cell*(R,s_1,s_2) to *cell*(R',s_1,s_2) for each $[s_1,s_2] \in \sigma(R) \cap \sigma(R')$, and connect each *cell*(R,s_1,s_2), $[s_1,s_2] \in \sigma(R) \setminus \sigma(R')$, if any, to each *cell*$(R',s'_1,s'_2)$, $[s'_1,s'_2] \in \sigma(R') \setminus \sigma(R)$, if any

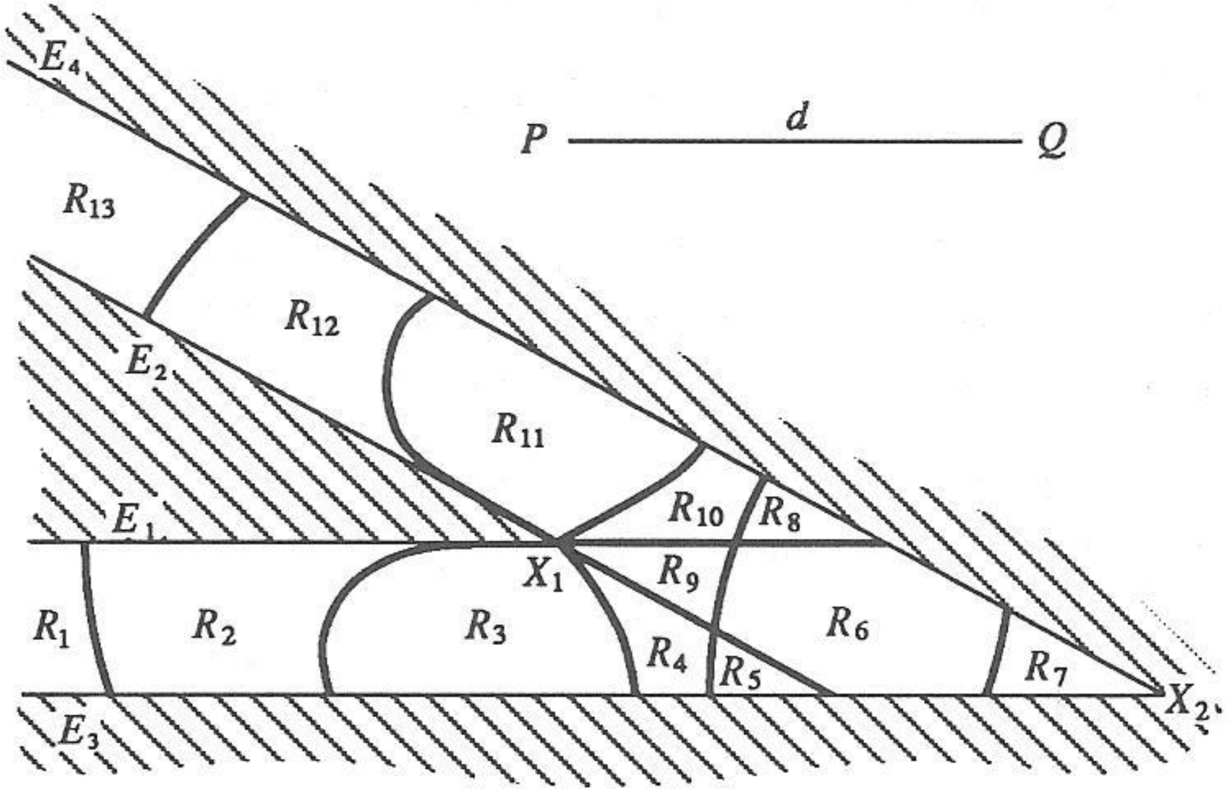


Figure 12. This figure shows the decomposition of a polygonal workspace into 13 noncritical regions. The robot can move from one end of the "corner" to the other, but it cannot make a full rotation in the corner. Thus, when the robot exits from the corner at one end, its orientation is determined by its orientation when it entered the corner at the other end.

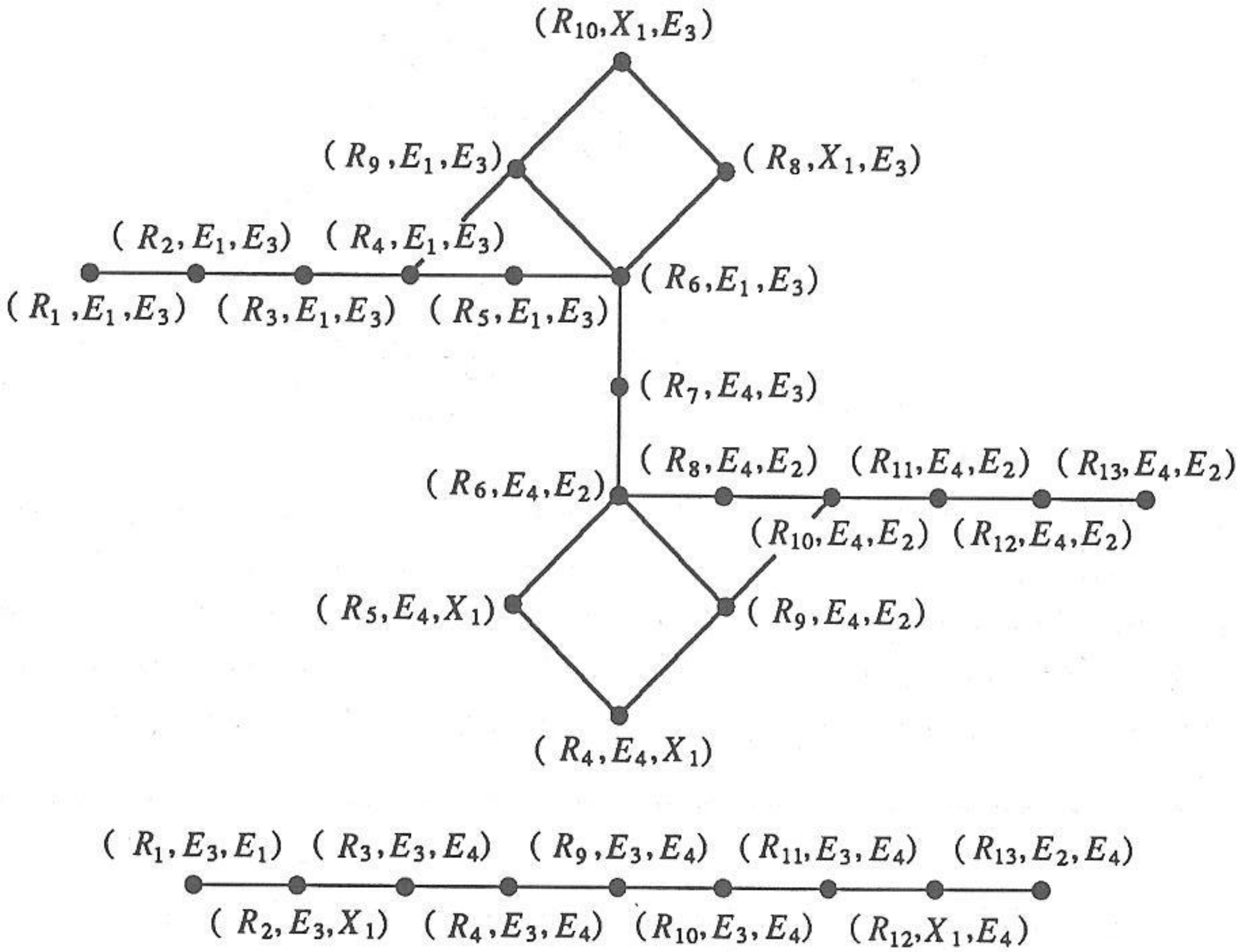


Figure 13. This figure shows the connectivity graph for the example of Figure 12. It consists of two connected components, hence verifying the legend of Figure 12.

Complexity

- The number of critical curves
- The number of noncritical regions
- The combinatorial complexity of a cylinder above a noncritical region
- The number of vertices and edges in the connectivity graph
- Total : $O(n^5)$

- Can this bound be achieved?

- Can we do better?

From Piano Movers I (rod) to Piano Movers II (general)

- Ingredient: Combinatorics and algebra
- In PM I we saw both of them
- The algebra aspect: resultants and more
- The combinatorial aspect: arrg of Δs, for example

Solution to general MOP with 2 DOFs using CGAL, following the separation of algebra and combinatorics (later)

References

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JC **Latombe** - 1991 - Springer

THE END