On the Complexity of the Free Space of a Disc Moving among Discs in the Plane

Let A be a disc robot moving in the plane among n disc obstacles $B_1, B_2, \ldots, B_n$. Let D_i denote the configuration-space obstacle induced by B_i. D_i is a disk whose radius is the sum of radii of B_i and of A. We are interested in the combinatorial complexity of the *free space* of this motion planning problem, which is the complement of the union of the expanded discs D_i. We will bound the number of vertices on the boundary of the free space.

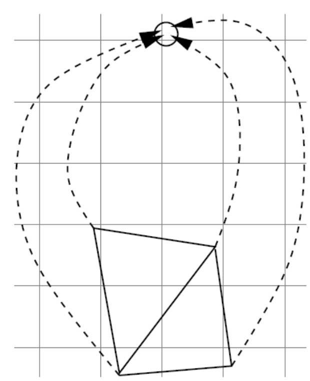
Theorem 1. *The number of vertices on the boundary of the union of $n \geq 3$ discs in the plane is at most $6n - 12$, and this bound is tight.*

Proof. We use the so-called *lifting transform* and lift our problem to the unit paraboloid $\mathcal{U} : z = x^2 + y^2$ in $\mathbb{R}^3$ as follows. Every point (x, y) in the plane is projected vertically onto $\mathcal{U}$, namely it is lifted to the point $(x, y, x^2 + y^2)$.

Let C_i denote the boundary of the disc D_i, and let γ_i denote the lifting of C_i to $\mathcal{U}$. The interesting observation is that γ_i lies on a plane. This is easy to see. If the circle C_i is centered at (a_i, b_i) and has radius r_i, its equation is

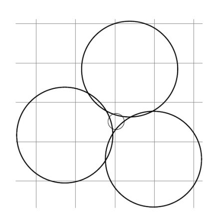
$$x^2 - 2a_i x + a_i^2 + y^2 - 2b_i y + b_i^2 = r_i^2.$$

Since γ_i lies on $\mathcal{U}$, we can substitute $x^2 + y^2$ by z in the equation and we get the equation of a plane, which we denote by H_i. The plane H_i cuts out of $\mathcal{U}$ exactly the forbidden region expressed by D_i. Let H_i^+ denote the halfspace bounded by H_i and above it. The (lifted) free space lies on $\mathcal{U} \cap H_i^+$. This is true for all the discs D_i. Let $\mathcal{P} := \cap_i H_i^+$. The lifted free space is $\mathcal{U} \cap \mathcal{P}$.



$\mathcal{P}$ is the intersection of halfspaces, namely it is a convex polyhedron, such that its boundary is an xy-monotone (the polyhedron is bounded from below only) piecewise-linear surface. To count the number of edges in $\mathcal{P}$ we project it onto the xy plane. In order to employ Euler's formula for planar graph, we first have to make the resulting graph finite. We do this by adding an artificial vertex far away from all other vertices, and connecting all the infinite edges to this vertex so that they do not cross one another; see the figure to the right. Notice that the number of edges and faces does not change. Let V, E and F denote the number of vertices, edges and faces in this graph respectively. By Eurler's formula $V - E + F = 2$. In our graph every vertex is incident to at least three edges, hence $V \leq 2E/3$. It follows that $E \leq 3F - 6 = 3n - 6$. In summary, the number of edges in $\mathcal{P}$ is at most $3n - 6$.

Every edge of $\mathcal{P}$ intersects $\mathcal{U}$ at most twice, because $\mathcal{U}$ is a convex surface. These intersection points are the projection of the vertices of the union of discs onto $\mathcal{U}$. Hence their number is at most $6n - 12$ as asserted.



This bound is tight. Consider three discs intersecting in pairs—these are the three big discs in the figure. There are six vertices on the boundary of their union. Now place one small disc such that it intersects each of the three big discs in two points, leaving three holes. This fourth disc adds six vertices. In each of these holes we can place the center of yet a smaller disc that will induce six new vertices and will leave three holes, and so on.

□

Remarks. (1) Theorem 1 implies that the complexity of the free space of the motion planning for a disc among discs is $O(n)$.

(2) Theorem 1 seems to be common knowledge and probably rediscovered several times.[1] It is however a special instance of a more general phenomenon. We can substitute the set of discs by a family of *pseudodiscs* and the result still holds. A set $\mathcal{R}$ of planar regions is called a set of pseudodiscs, if each region is bounded by a closed Jordan curve, and for each pair of regions $R_1, R_2 \in \mathcal{R}$ such that $R_1 \cap R_2 \neq \emptyset$, $\partial R_1 \cap \partial R_2$ consists of two points.

In class we will show that for a set of polygonal pseudodiscs with a total of n vertices, the boundary of their union has complexity $O(n)$ [1, Chapter 13].

(3) The most general result, due to Kedem et al. [2], pertains to arbitrary collections of pseudodiscs and asserts that the number of intersection points on the boundary of n pseudodiscs is at most $\max(2, 6n - 12)$. The proof of this general case is considerably more difficult.

References

[1] M. de Berg, M. van Kreveld, M. H. Overmars, and O. Schwarzkopf. *Computational Geometry: Algorithms and Applications.* Berlin, Germany, 2nd edition, 2000.

[2] K. Kedem, R. Livne, J. Pach, and M. Sharir. On the union of Jordan regions and collision-free translational motion amidst polygonal obstacles. *Discrete Comput. Geom.*, 1:59–71, 1986.

[1] Now that this is on the web, if you know its origin, let me know.