Algorithmic Robotics and Motion Planning

The Roomba in the café

Combinatorics and algorithms

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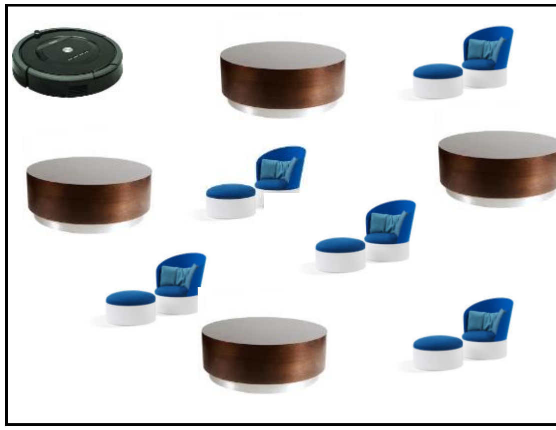
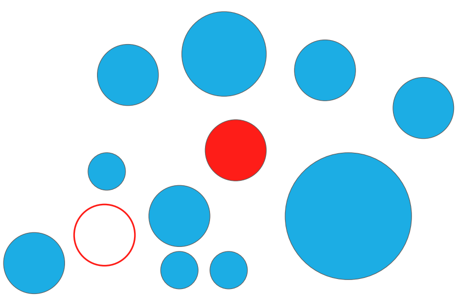


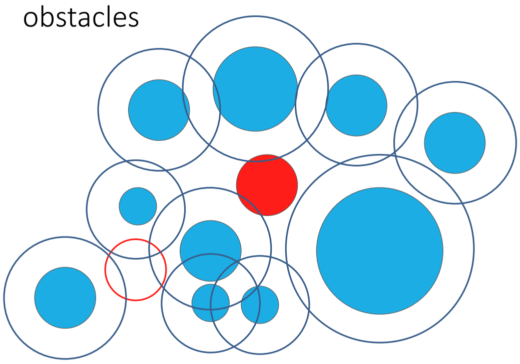
Moving a disc among discs



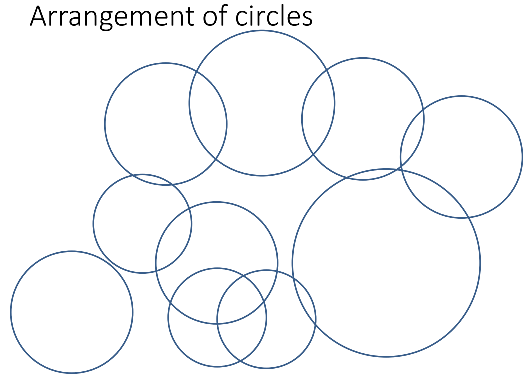
Outline

- the C-space
- combinatorial complexity
- representation
- algorithm
- algebra

Moving a disc among discs: C-obstacles



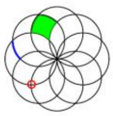
Arrangement of circles



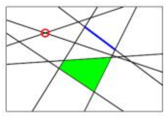
Arrangements (take I)

Definition (Arrangement)

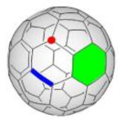
Given a collection $\mathscr{C}$ of curves on a surface, the arrangement $\mathscr{A}(\mathscr{C})$ is the partition of the surface into vertices, edges and faces induced by the curves of $\mathscr{C}$.



An arrangement of circles in the plane.

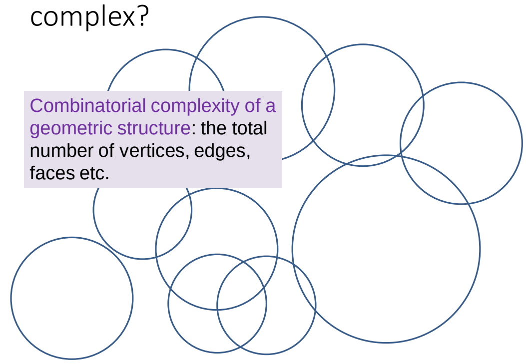


An arrangement of lines in the plane.



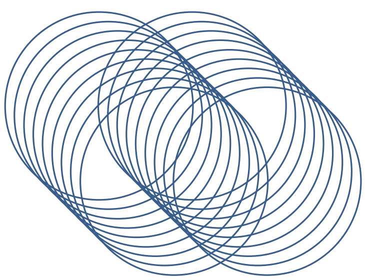
An arrangement of great-circle arcs on a sphere.

Arrangement of circles: how complex?

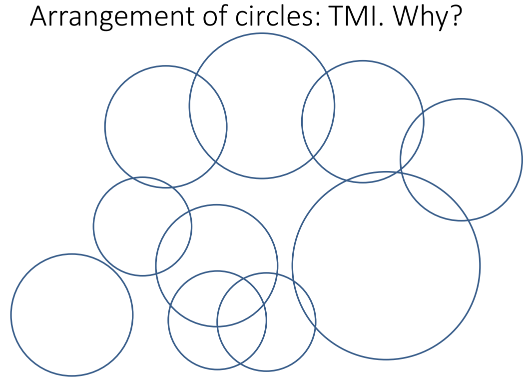


Combinatorial complexity of a geometric structure: the total number of vertices, edges, faces etc.

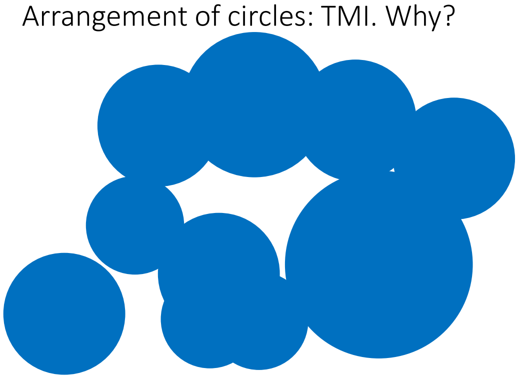
Arrangement of circles: how complex?



Arrangement of circles: TMI. Why?



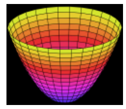
Arrangement of circles: TMI. Why?



Combinatorial analysis

- n – the number of obstacle discs
- arrangement of n circles
- the union of n discs
  - the lifting transform
  - the complexity of a 3-poytope

The lifting transform



[wikipedia]

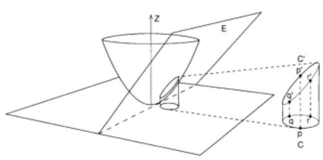
- the lifting transform maps points in R^d to objects (points or hyperplanes) in R^{d+1}
- we will focus on the plane, and the vertical projection of planar points onto the *unit paraboloid* U in R^3:
  $U: z = x^2 + y^2$
- vertical cross-sections of U are parabolas, horizontal cross-sections are circles
- $LT: p(x, y) \longmapsto \hat{p}(x, y, x^2 + y^2)$

Lifting a circle

- $LT: p(x, y) \longmapsto \hat{p}(x, y, x^2 + y^2)$
- $C(a, b, r)$ is a circle in the plane with center at (a, b) and radius r
- $LT: C(a, b, r) \longmapsto$?
- $C:\ (x - a)^2 + (y - b)^2 = r^2$
- $C:\ x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$
- $\hat{C}$ is on U, therefore in $\hat{C}$ we can replace $x^2 + y^2$ by z, to obtain
- $z = 2ax + 2by - (a^2 + b^2 - r^2)$

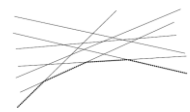
Lifting a circle, cont'd

- $z = 2ax + 2by - (a^2 + b^2 - r^2)$
- the lifted circle $\widehat{C}$ resides on a plane!

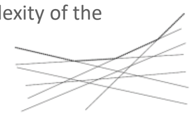


[Aurenhammer and Klein]

Envelopes

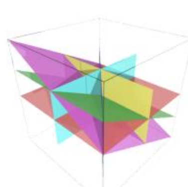


- arrg of n lines
- what is the shape below the lower envelope?
- what is the exact maximum complexity of the envelope?
- what is the shape above the upper envelope?
- what is the exact maximum complexity of the envelope?



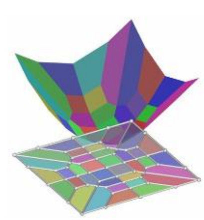
Arrangements of planes and their lower envelope

- arrg of n planes, H
- the upper and lower envelope: shape and complexity



[wikipedia]

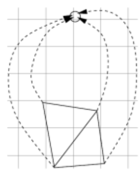
Degenerate upper envelope of planes and its minimization diagram



- we assume henceforth general position

The complexity of the upper envelope of planes

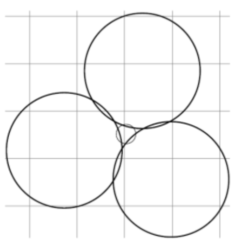
- Euler's formula $V - E + F = 2$
- Each vertex has at least 3 incident edges, $V \leq 2E/3$
- Together $E \leq 3F - 6 \leq 3n - 6$



The number of vertices on the boundary of the free space

- U intersects each edge of the upper envelope at most twice: these are the vertices of the free space
- Their number is therefore at most $6n - 12$

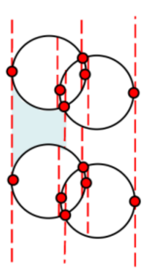
Combinatorial analysis, lower bound



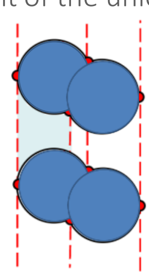
Algorithms for computing the union of discs

- representation: DCEL
- Algorithm I: divide and conquer using plane sweep in the merge step
- Algorithm II: mimicking the proof of the combinatorial bound

Vertical decomposition of arrgs of circles



Vertical decomposition of the complement of the union of discs



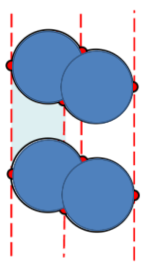
Algorithms for solving the Roomba MP problem

- augment the DCEL with vertical decomposition
- build a connectivity graph (CG) over the augmented DCEL:
  - a node for every free trapezoid
  - an edge between two trapezoids that share a vertical wall
- ../..

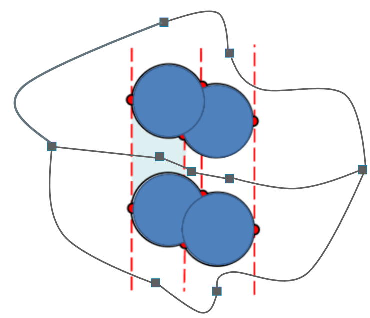
Algorithms for solving the Roomba MP problem, cont'd

- find the cells that contain the start and goal positions
- search in the CG for a path between the start node to the goal node
- transform the path in the graph into a collision-free path in the plane

Reminder: Vertical decomposition of the complement of the union of discs



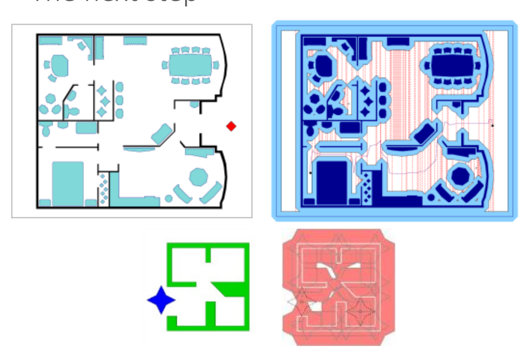
The connectivity graph



Reference

- Writeup (combinatorial analysis) on the course's website

The next step



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