Separating a Polyhedron from Its Single Part Mold: Optimal Algorithms

Computational Geometry

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Overview

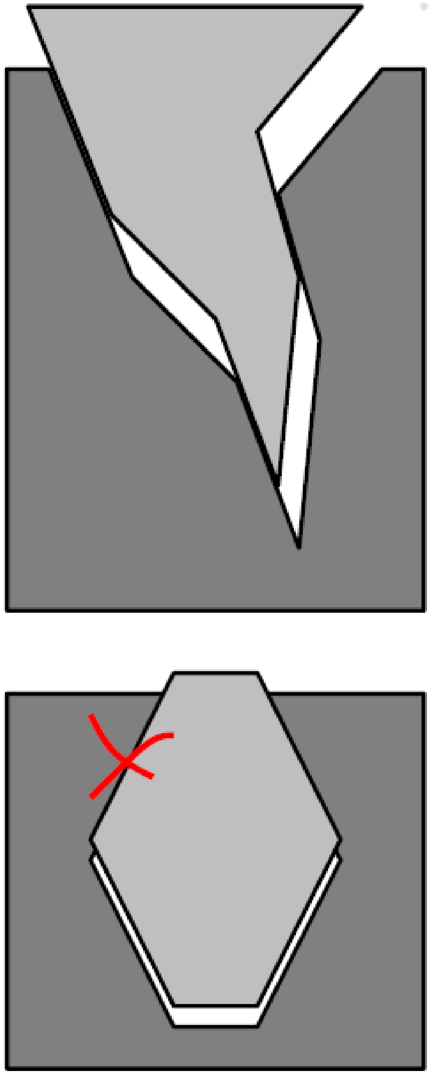
- the casting problem, a reminder
- Helly’s theorem
- arrangements on the sphere
- alternative approach to the casting problem

Credits

- CGAA: some figures are taken from Computational Geometry Algorithms and Applications by de Berg et al
- the original figures are available at the book's site: www.cs.uu.nl/geobook/
- L5vK: Lecture 5 in Computational Geometry, Casting a polyhedron, by Marc van Kreveld

The casting problem

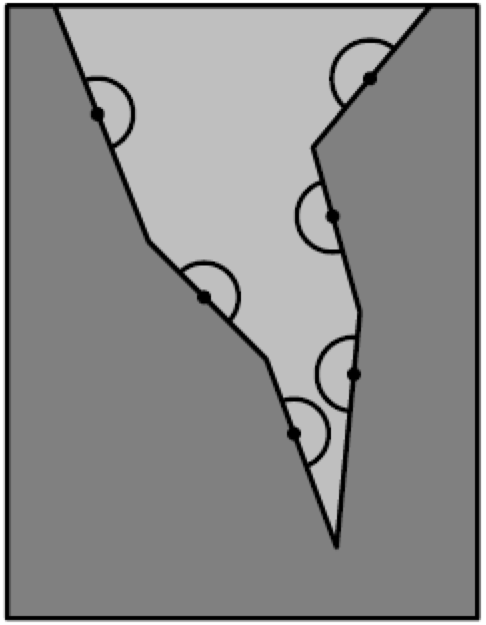
First the 2D version: can we remove a 2D polygon from a mold?



[L5vK]

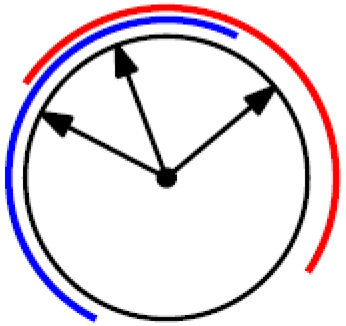
A polygon can be removed from its cast *by a single translation* if and only if there is a direction so that every polygon edge does not cross the adjacent mold edge

Sequences of translations do not help; we would not be able to construct more shapes than by a single translation



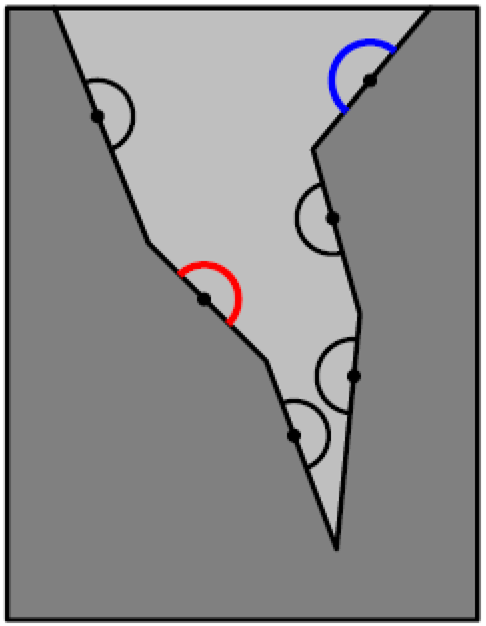
[L5vK]

We need a representation of directions in 2D



Every polygon edge requires the removal direction to be in a semi-circle

⇒ compute the common intersection of a set of circular intervals (semi-circles)

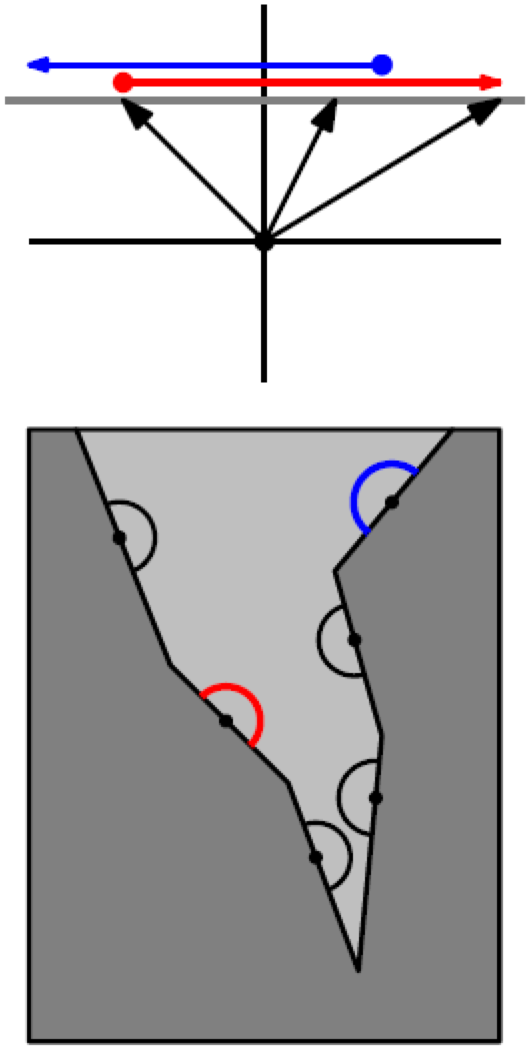


[L5vK]

We only need to represent upward directions: we can use points on the line $y = 1$

Every polygon edge requires the removal direction to be in a half-line

$\Rightarrow$ compute the common intersection of a set of half-lines in 1D



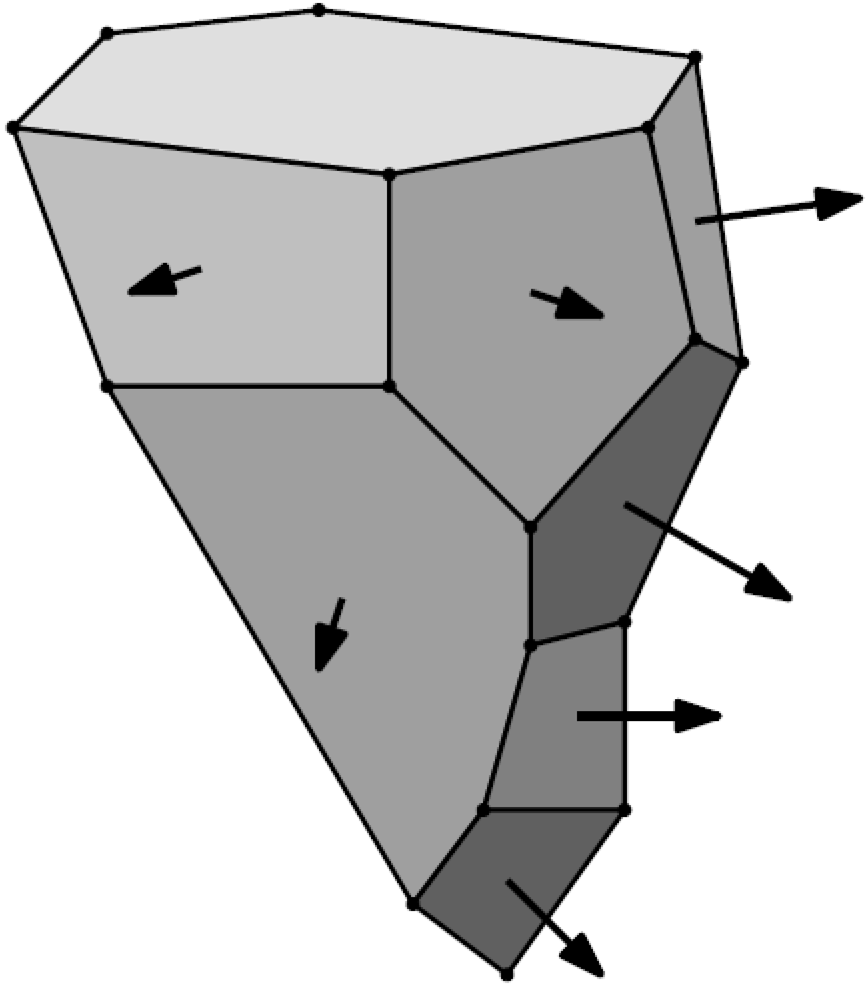
[L5vK]

In 3D, for a candidate top facet

Consider the outward normal vectors of all facets

An allowed removal direction must make an angle of at least $\pi/2$ with every facet (except the topmost one)

$\Rightarrow$ every facet in 3D makes a half-plane in $z = 1$ invalid

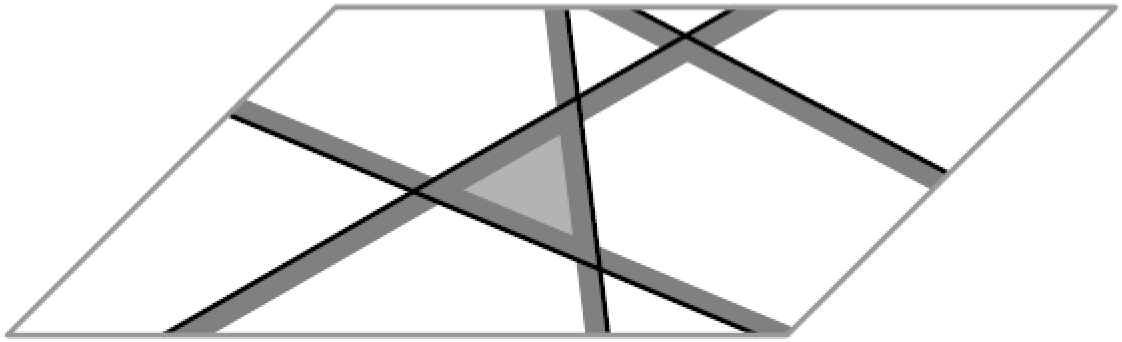


[L5vK]

For every candidate top facet

We get: common intersection of half-planes in the plane

The problem of deciding castability of a polyhedron with n facets, with a given top facet, where the polyhedron must be removed from the cast by a single translation, can be solved by computing the common intersection of $n-1$ half-planes



[L5vK]

The (previous) solution

- *all* directions for every valid top facet in $O(n^2 \log n)$ time: intersection of half-planes per candidate top facet
- *one* direction for every valid top facet in $O(n^2)$ time: Linear Programming per candidate top facet

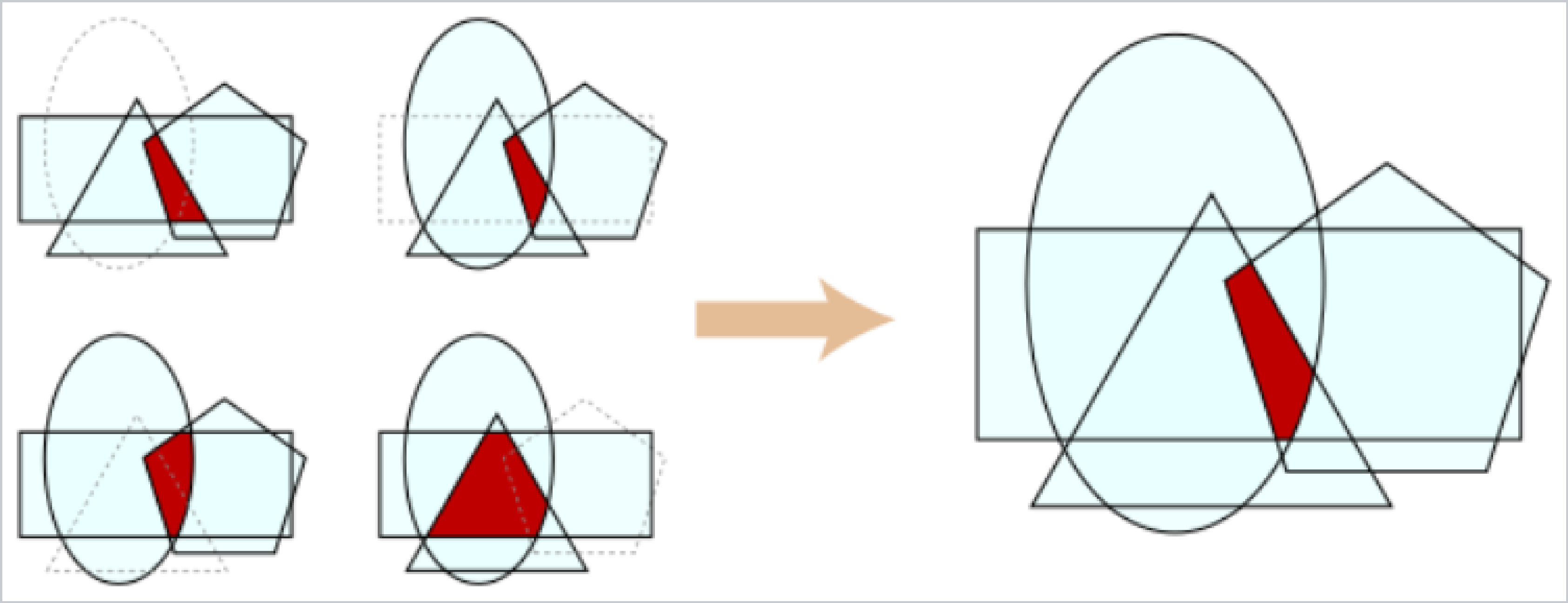
- can we do better?

Helly’s theorem

Statement

- Let $X_1, \ldots, X_n$ be a finite collection of convex subsets of R^d, with $n > d + 1$. If the intersection of every $d + 1$ of these sets is nonempty, then the whole collection has a nonempty intersection.

In the plane



Helly's theorem for the Euclidean plane: if a family of convex sets has a nonempty intersection for every triple of sets, then the whole family has a nonempty intersection.

[wikipedia]

Arrangements on the sphere

Arrangements of great circles

- n great circles
- the arrangement has at most $n(n-1)$ vertices, $2n(n-1)$ edges, and $n^2 - n + 2$ faces
- the central projection of the arrangement on a hemisphere onto a tangent plane is an arrangement of lines

A different approach to the casting problem

Outline and notation

- we consider all valid top facets simultaneously
- we will use the entire sphere of directions S^2
- (for convenience of arguing we will project it onto the plane $z = 1$, as before, but also onto the plane $z = -1$, and handle the equator separately)
- fix the orientation of our polyhedron P, arbitrarily
- $F_1, F_2, \dots, F_n$: the facets of P
- $\nu(F_i)$: the normal of F_i pointing *into* the polytope P

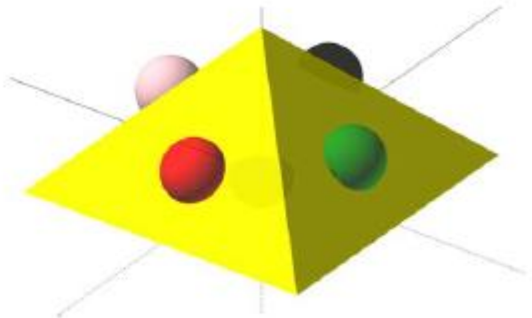
The pair $(F_i, \vec{d})$

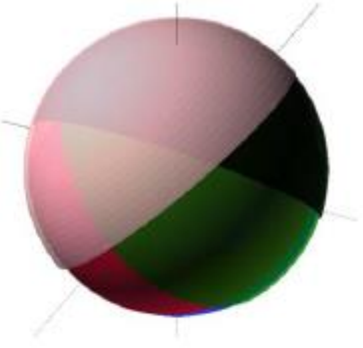
- $(F_i, \vec{d})$: $\vec{d}$ represents a pullout direction when F_i is the top facet of the mold – should be interpreted as follows: P is rotated such that F_i becomes the top facet and $\vec{d}$ is rotated accordingly
- the key observation:

  $(F_i, \vec{d})$ represents a *valid* mold and pullout direction iff:
  - $\vec{d} \cdot \nu(F_i) < 0$
  - $\forall j \neq i,\ \vec{d} \cdot \nu(F_j) \geq 0$
- (we proved a similar claim for the original solution)

The hemispheres induced by F_i

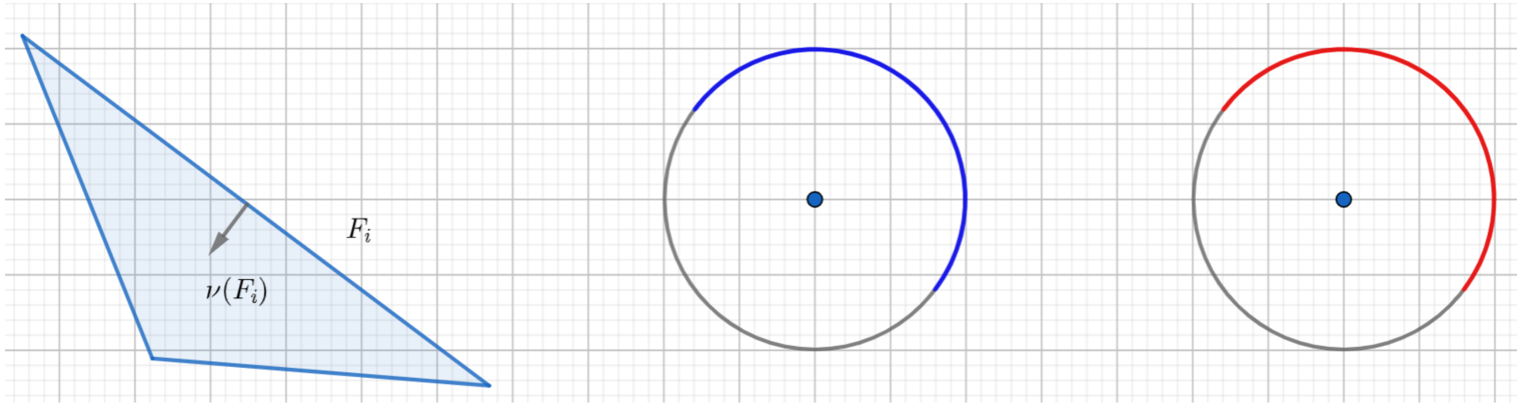
- every facet F_i of P splits S^2 into the disjoint union of: (i) a closed hemisphere of forbidden pullout directions, and (ii) the complementary open hemisphere of potential pullout directions, when F_i is the top facet
- $h_i := h(F_i)$ is the closed hemisphere $\vec{d} \cdot \nu(F_i) \geq 0$
- $\bar{h}_i$ is the complement open hemisphere $\vec{d} \cdot \nu(F_i) < 0$





The different roles of $\bar{h}_i$

- when we consider F_i to be the top facet, $\bar{h}_i$ represents the valid pullout directions
- when we consider F_j to be the top facet for $j \neq i$, the same $\bar{h}_i$ represents forbidden pullout directions



- hence we look for points on S^2 that are covered by exactly one hemisphere $\bar{h}_i$

Arrangements on the sphere

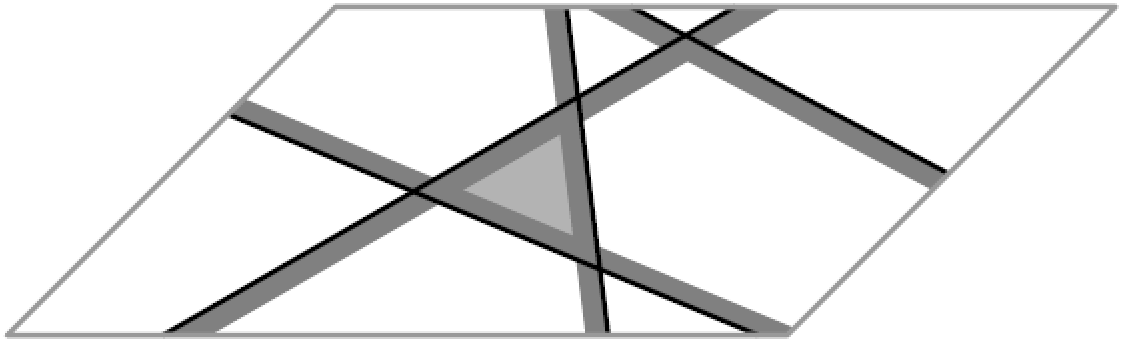
- the n boundary great circles of the h_i's partition S^2 into faces
- this is an arrangement of great circles (which, as mentioned, behaves similarly to an arrangement of lines in the plane)
- the constraints $\vec{d} \cdot \nu(F_i) < 0$ (namely the hemispheres $\bar{h}_i$) are the same in each face of the arrangement!
- we look for a face of the arrangement that is covered by exactly one constraint $\bar{h}_i$: it will represent valid directions for F_i as a top facet and no other facet precluding its validity
- can easily be found in $O(n^2)$ time

The depth of a face in an arrangement of hemispheres

- the depth of a face: the number of $\bar{h}_i$'s that cover it
- we look for a face of the arrangement of depth 1, covered by exactly one constraint $\bar{h}_i$: it will represent valid directions for F_i as a top facet and no other facet precluding its validity
- it will be more convenient to transform this arrangement to planar arrangements (as promised): we will project the arrangement onto the plane $z = 1$ and the plane $z = -1$, and handle the equator separately
- on each plane we now have an arrangement of half-planes
- we focus on $z = 1$

Covering set

- we will look for a small set of candidates to be valid top facets and then check each candidate
- *covering set*: Let B be a set of regions in the plane, whose union covers the entire plane. A subset $S \subseteq B$ is a covering set if the union of regions in S covers the entire plane.



Covering

- Claim: For any polyhedron P, the full set of $\bar{h}_i$'s cover the entire S^2 .
- Proof: Consider an interior point $p \in P$. For every direction $\vec{d}$, the ray from p in direction $\vec{d}$ will hit a boundary facet from inside P.

Lemma

- Let B be a set of half-planes in the plane, whose union covers the entire plane. Then, there is a covering set $S \subseteq B, |S| = 3$.
- Proof:

  $\bigcup_{b \in B} b = R^2 \Rightarrow$

  $\bigcap_{b \in B} \bar{b} = \emptyset \Rightarrow$

  $\exists b_i, b_j, b_k \in B$ s.t. $\bar{b}_i \cap \bar{b}_j \cap \bar{b}_k = \emptyset$

  (otherwise, by Helly, $\bigcap_{b \in B} \bar{b} \neq \emptyset$) $\Rightarrow$

  $b_i \cup b_j \cup b_k = R^2$

  QED

Finding the covering set

- we look for $\bar{h}_i, \bar{h}_j, \bar{h}_k$ such that $\bar{h}_i \cup \bar{h}_j \cup \bar{h}_k = R^2$
- readymade procedure: find h_i, h_j, h_k such that $h_i \cap h_j \cap h_k = \emptyset$
- where do we have such a procedure?
- when we find that a 2D linear program is infeasible
- can be determined in linear time

skipping various details …

Results

- $O(n \log n)$ time algorithm to find all possible pullout directions for all valid top facets
- this is optimal in the worst case [Geft]
- $O(n)$ time algorithm to find one pullout direction for each possible valid top facets
- $O(n)$ time algorithm to find all possible pullout direction for all valid top facets when P is *convex*
- for any polytope, there are at most six valid top facets and this bound is tight: there are polytopes with six valid top facets

For more details

Prosenjit Bose, Dan Halperin, Shahar Shamai:
On the separation of a polyhedron from its single-part mold. *CASE 2017: 61-66*

THE END

[Jeb Gaither, CGAL arrangements]