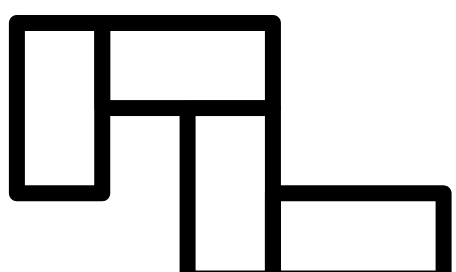
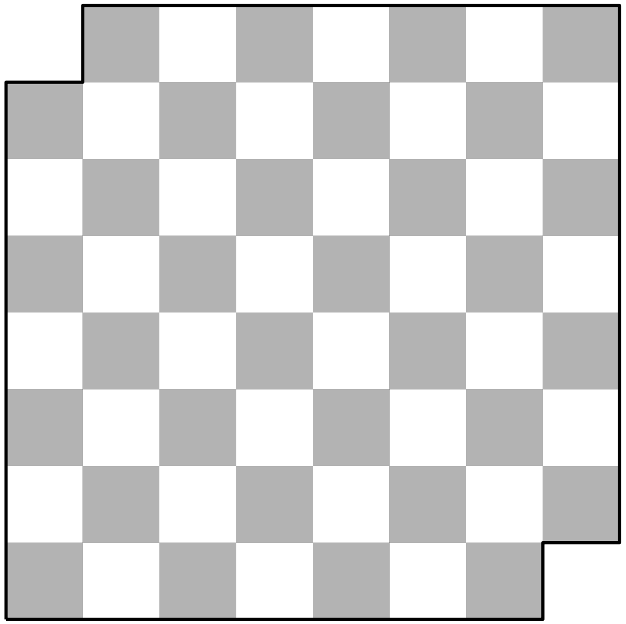
Tiling with Squares and Packing Dominos in Polynomial Time

Anders Aamand, <u>Mikkel Abrahamsen</u>, Thomas D. Ahle, Peter M. R. Rasmussen



Max Black, 1946: Two diagonally opposite corners have been removed from a chessboard. Can 31 1×2 dominos be placed to cover the remaining squares?



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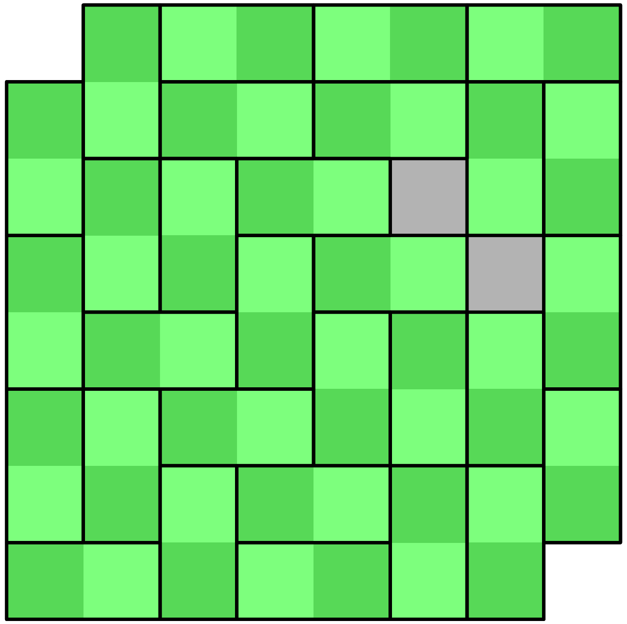
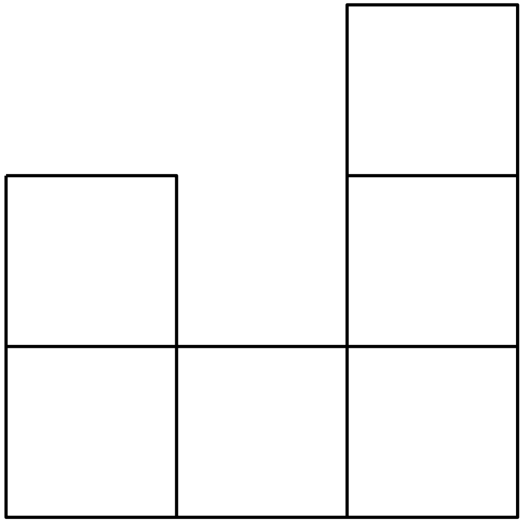


International Mathematical Olympiad 2004:
For which m and n can an $m \times n$ rectangle be tiled with 'hooks' of the following type:



Motivation

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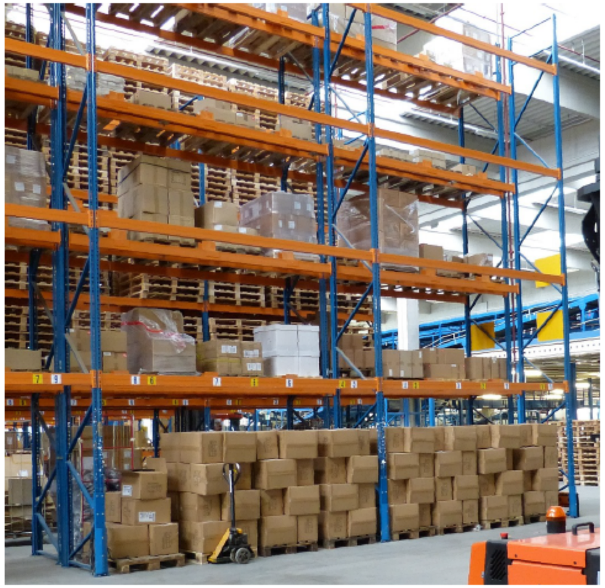

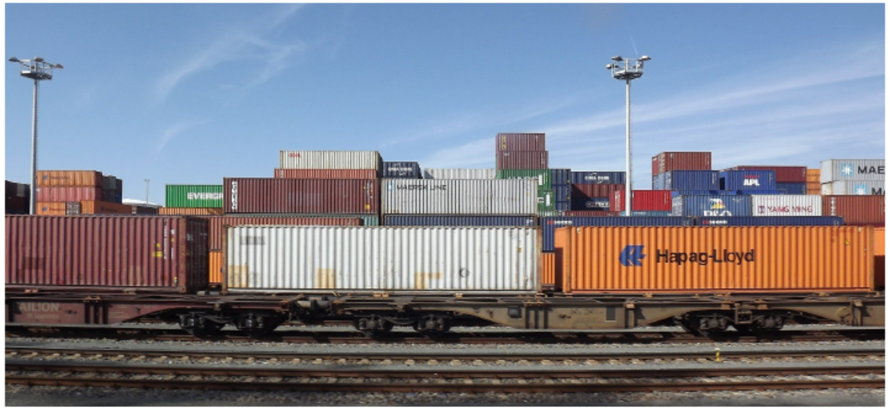

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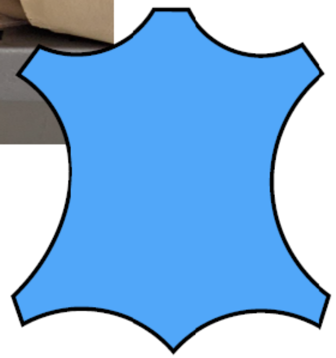
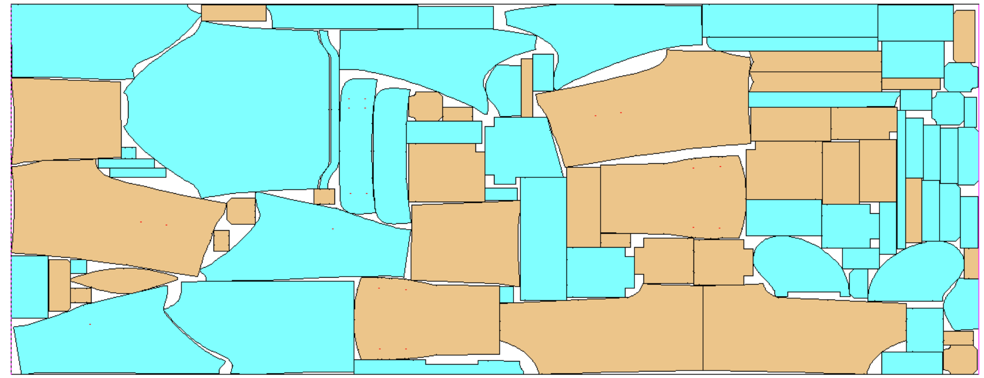
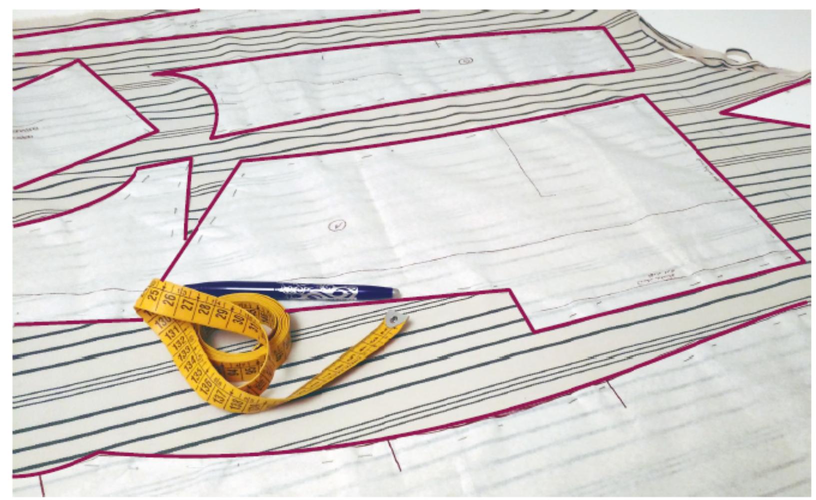
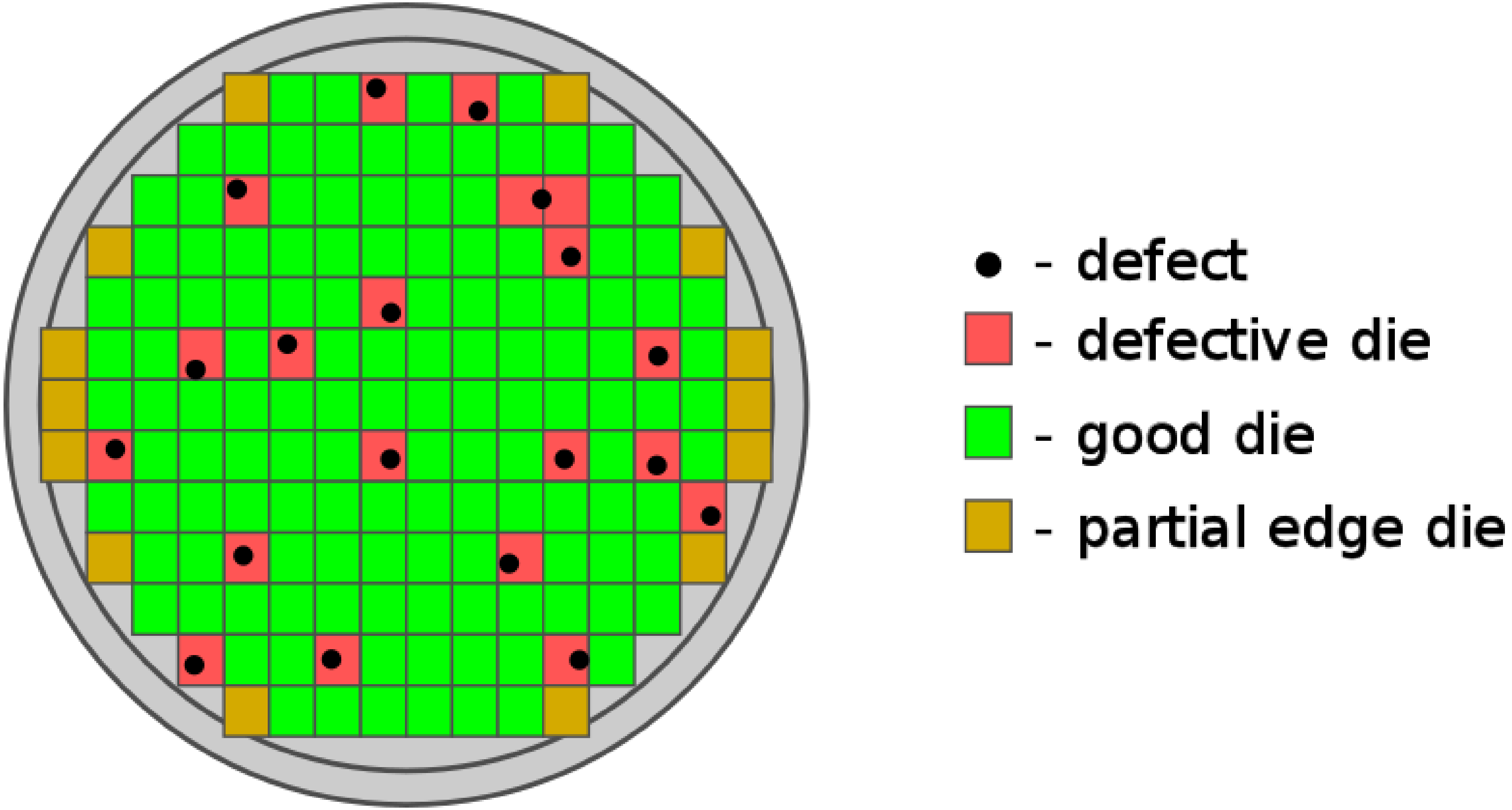






Motivation of domino packing



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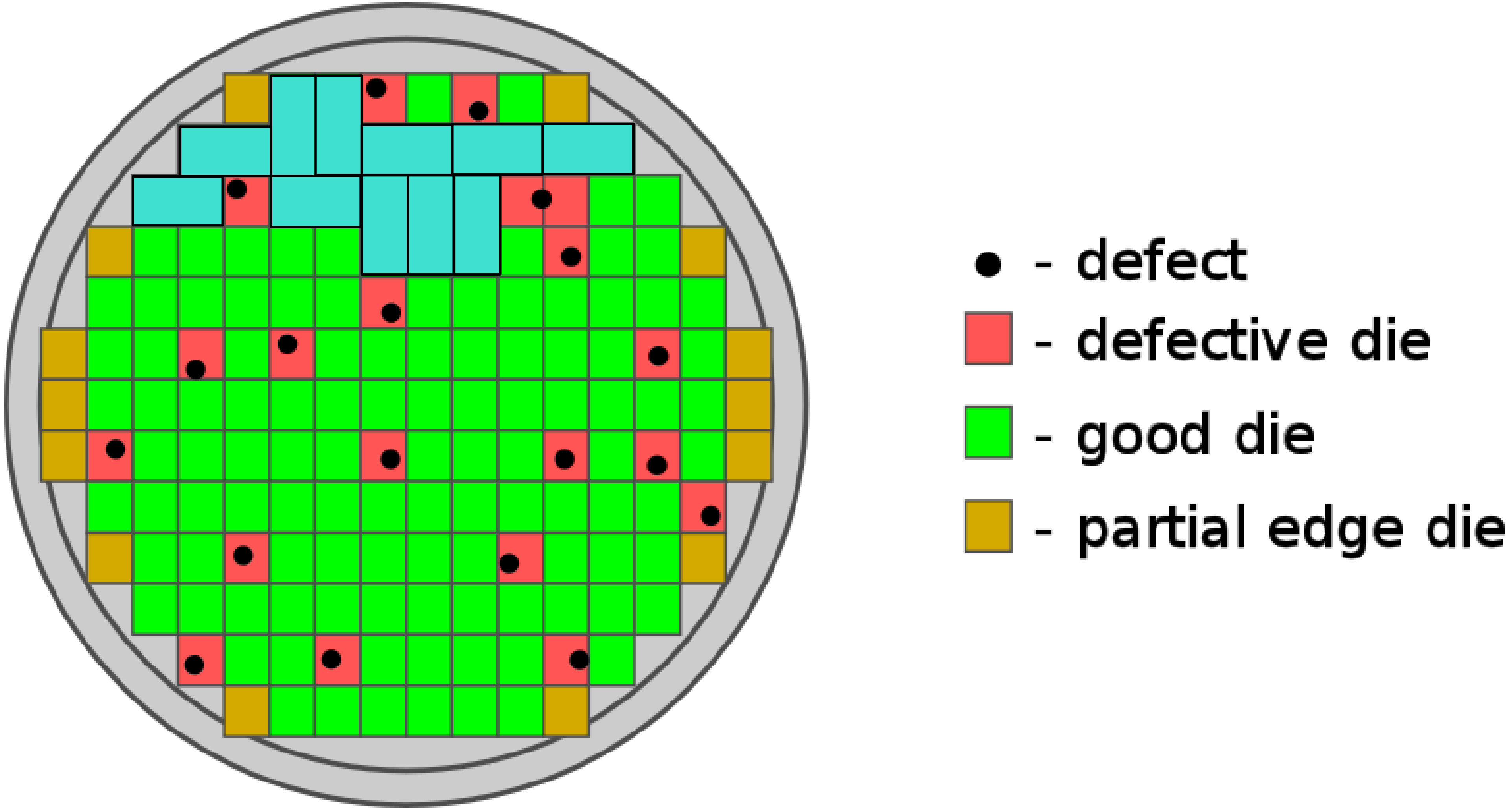
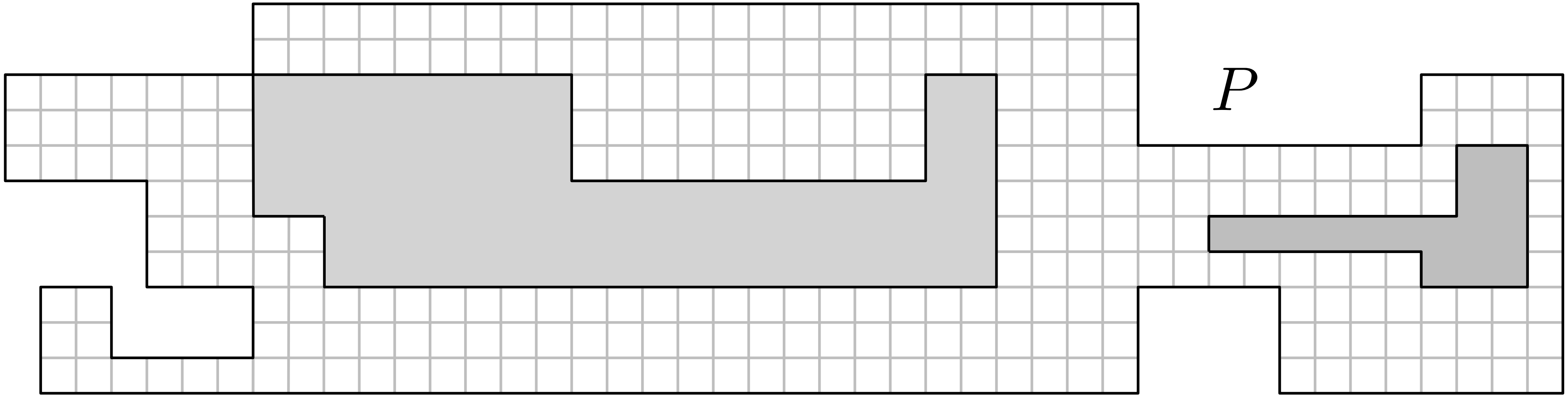


Polyomino: A polygonal region in the plane with axis-parallel edges and corners of integral coordinates.



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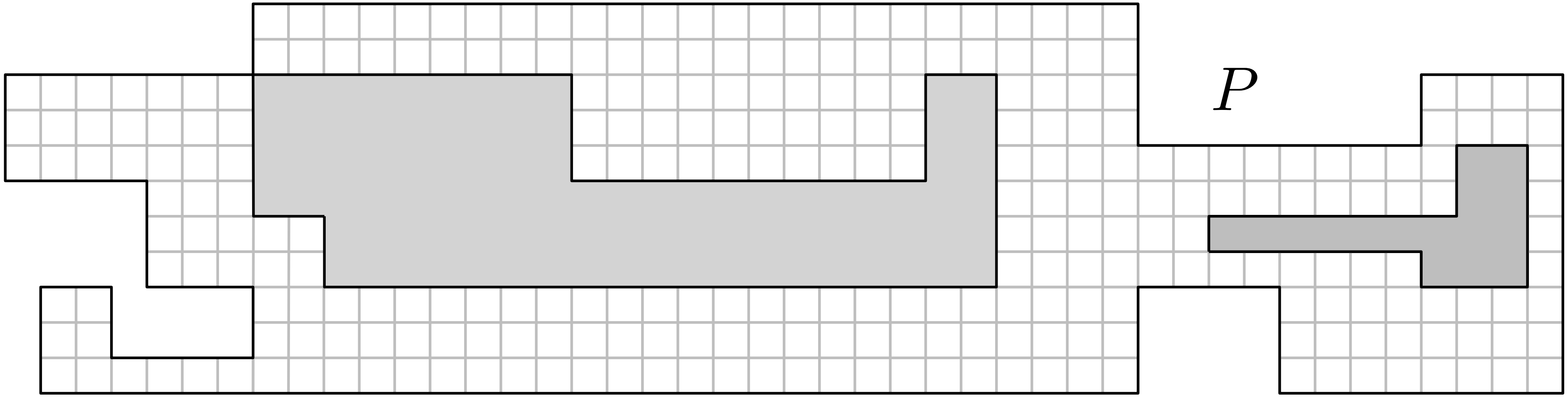
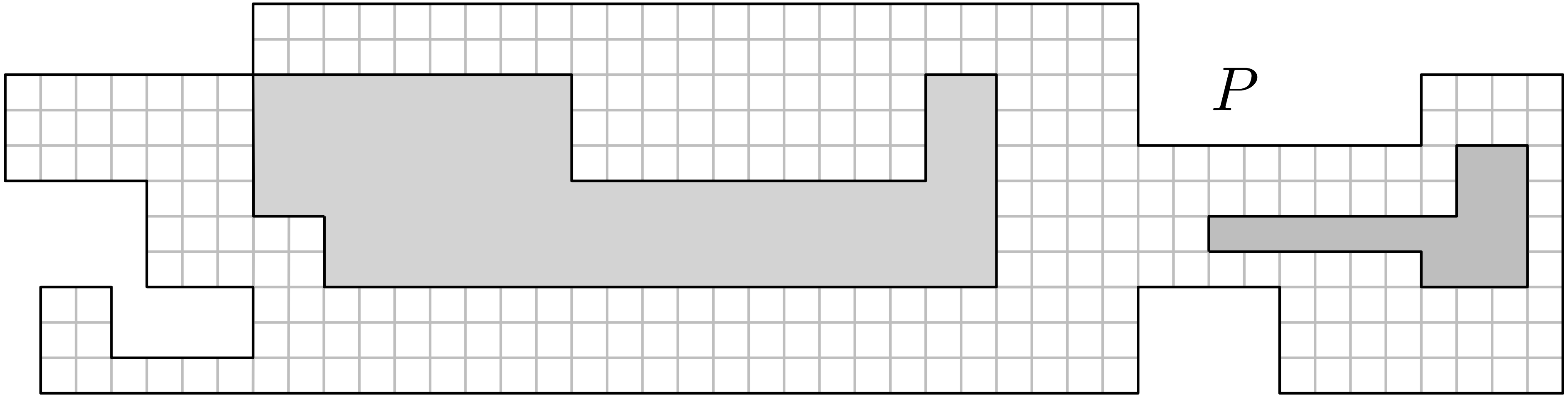


Tiling: Can a given large polyomino P be tiled with copies of a given small polyomino Q?

Polyomino: A polygonal region in the plane with axis-parallel edges and corners of integral coordinates.



Tiling: Can a given large polyomino P be tiled with copies of a given small polyomino Q?

Packing: How many non-overlapping copies of Q can be fit inside P?

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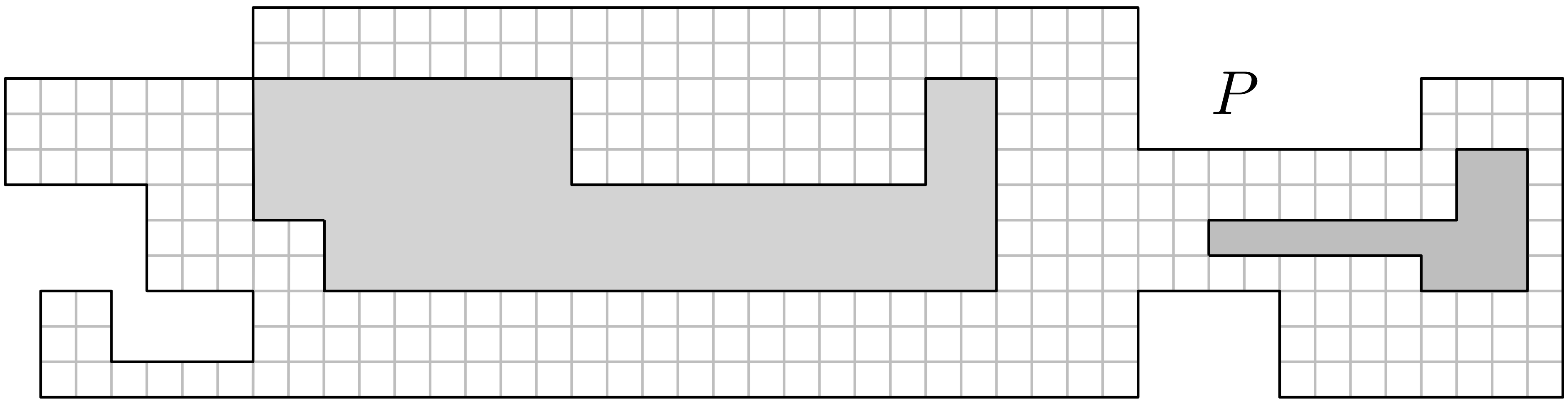
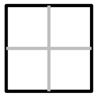
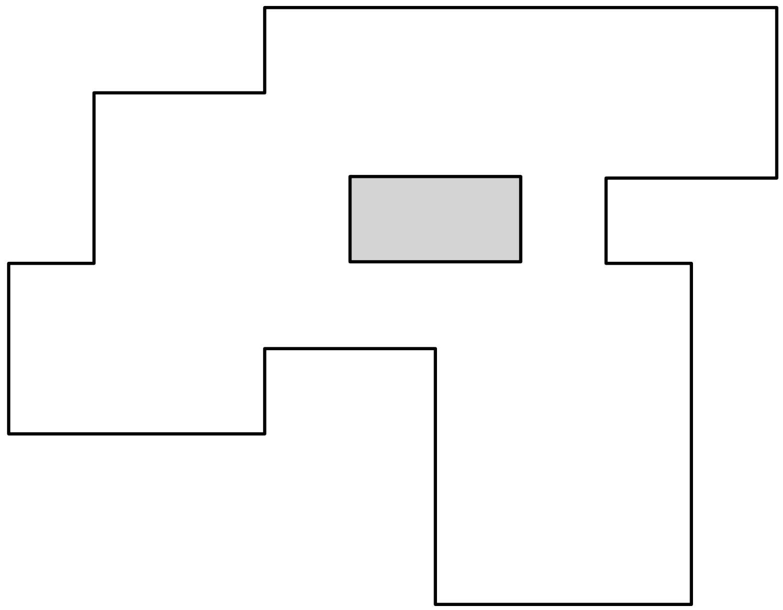


Tiling: Can a given large polyomino P be tiled with copies of a given small polyomino Q?

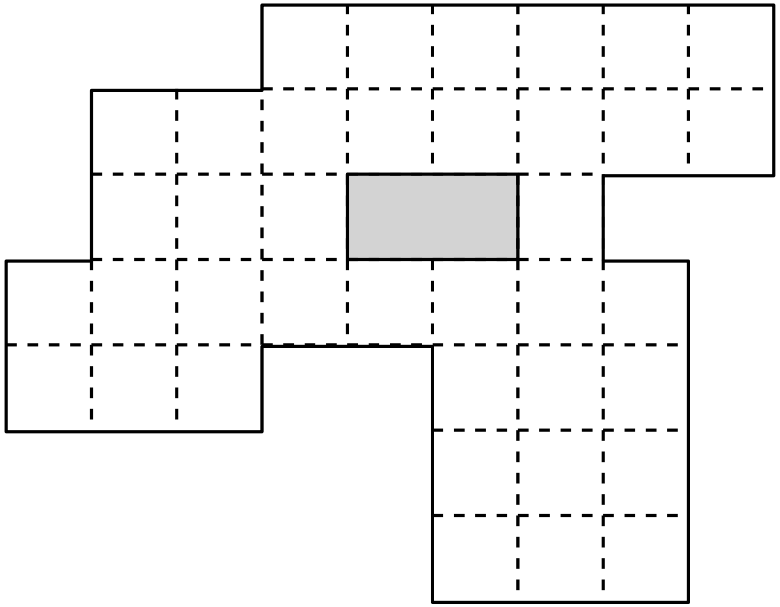
Packing: How many non-overlapping copies of Q can be fit inside P?

Our paper: $Q \in \{$  ,  $\}$

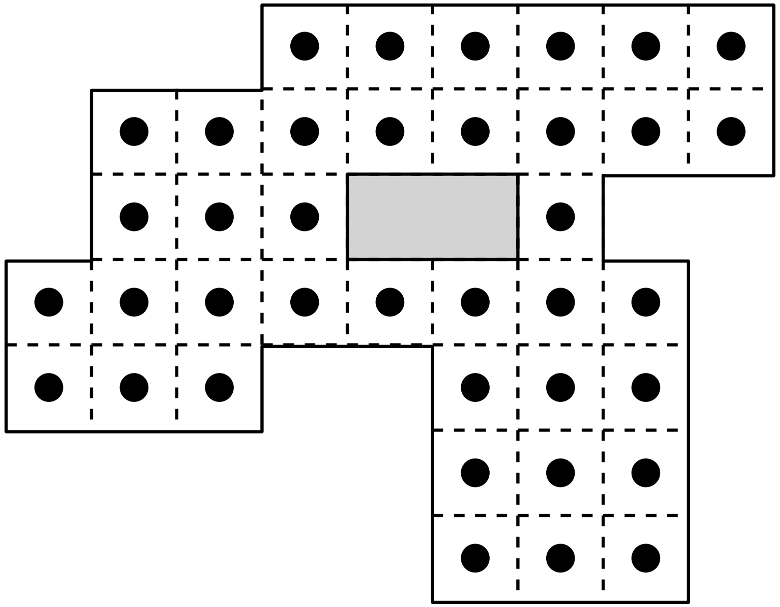
Representing a polyomino



Representing a polyomino



Representing a polyomino



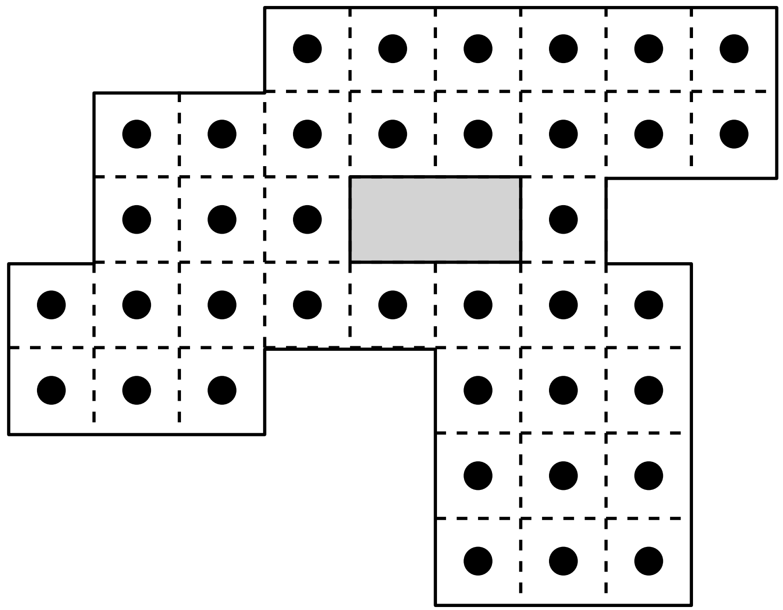
Usual way:

Store coordinates of each cell:

[• , • , • , • , • , • , . . .]

Area representation

Representing a polyomino



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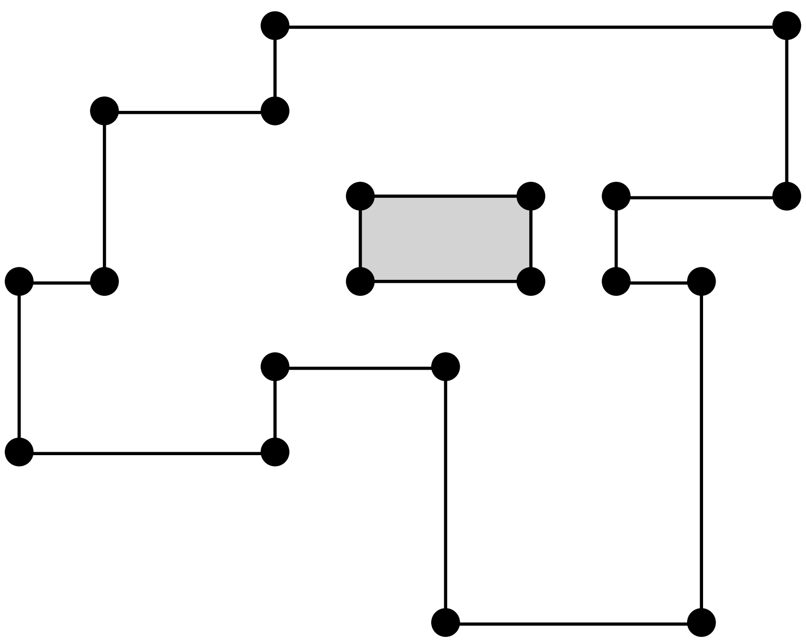
Area representation

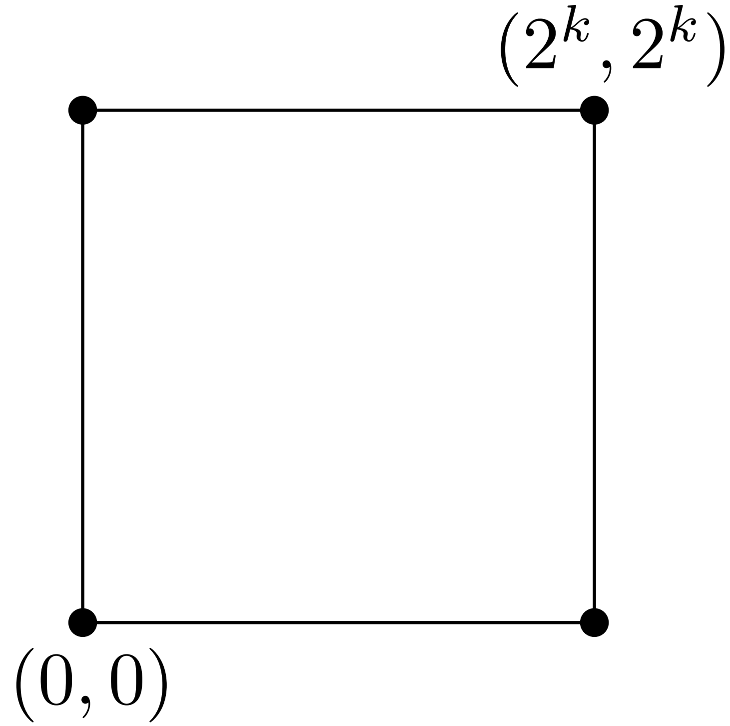


Compact way:

Store coordinates of corners.

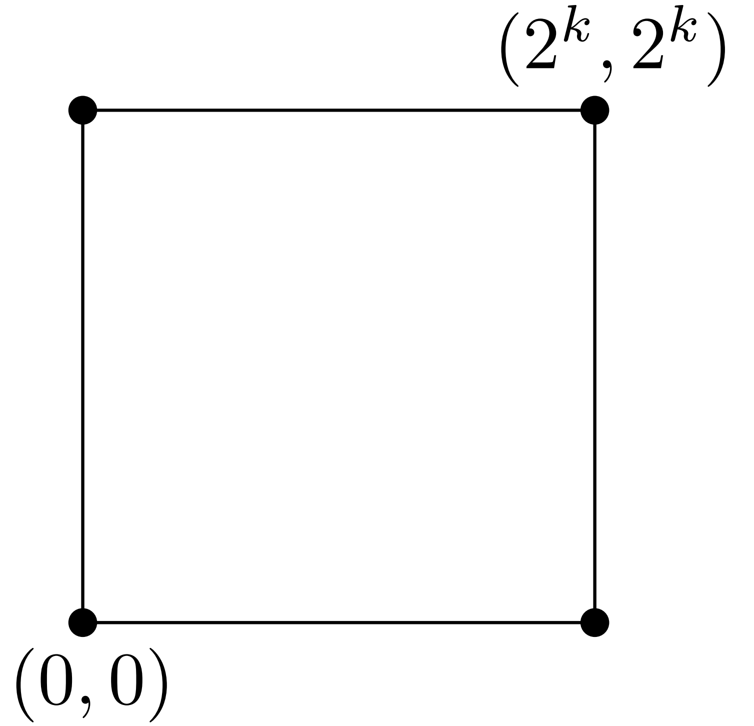
Corner representation

Example

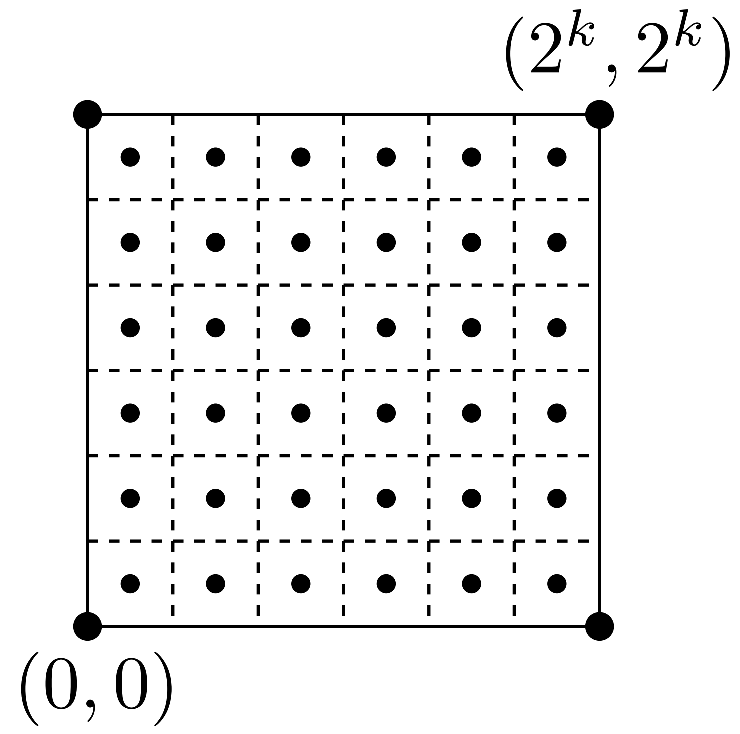


Example

Corner representation:
$[(0,0),(2^k,0),(2^k,2^k),(0,2^k)]$



Example



Corner representation:
$[(0,0),(2^k,0),(2^k,2^k),(0,2^k)]$

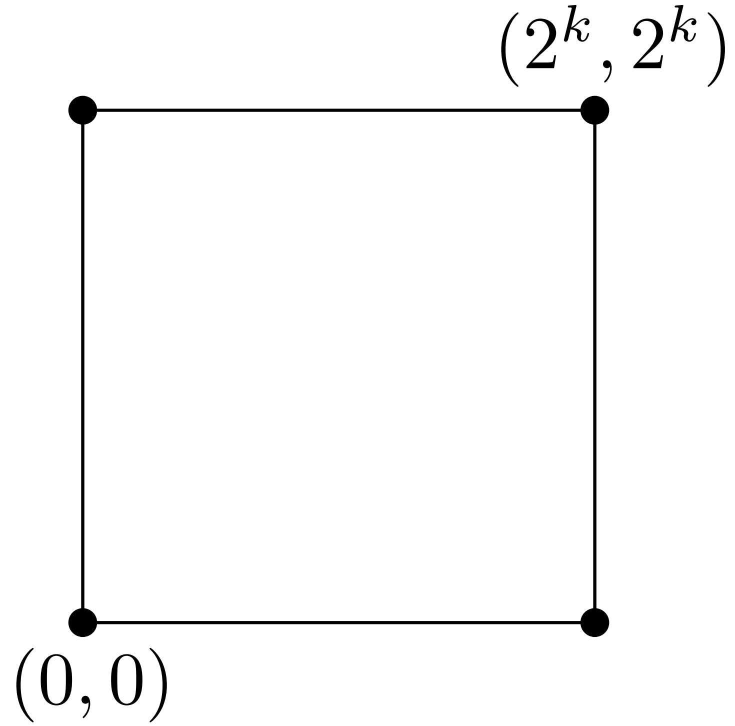
Area representation:
$[(0,0),(1,0),(2,0),\ldots,(2^k,0),$
$(0,1),(1,1),(2,1),\ldots,(2^k,1),$
$\vdots$
$(0,2^k),(1,2^k),(2,2^k),\ldots,(2^k,2^k)]$

Goal

Known algorithms:

Assume area representation $\Rightarrow$

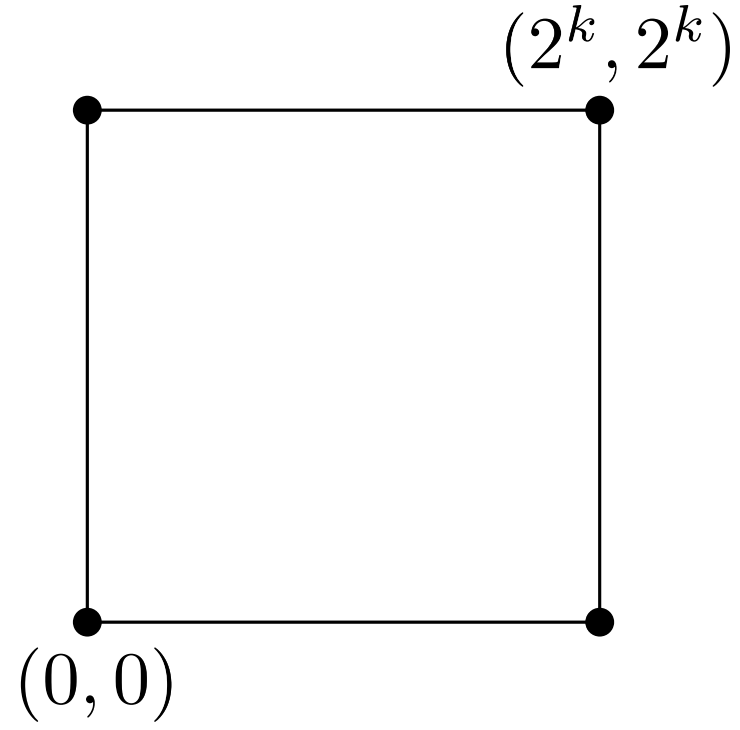
Time polynomial in the area.



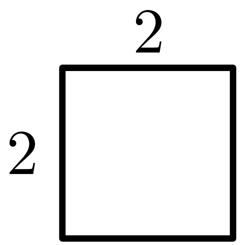
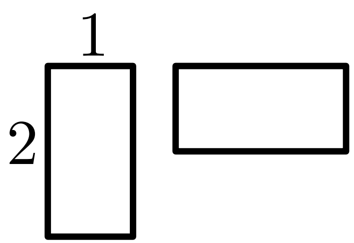
Goal

Known algorithms:
Assume area representation $\Rightarrow$
Time polynomial in the area.

Goal:
Assume corner representation.
Find algorithms with running time $O(\text{poly}(n))$.
n: the number of corners.



Results

Shapes	Tiling	Packing
	?	NP-complete
	?	?

Results

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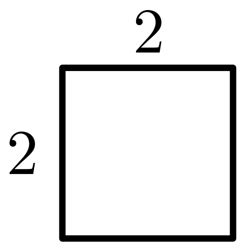
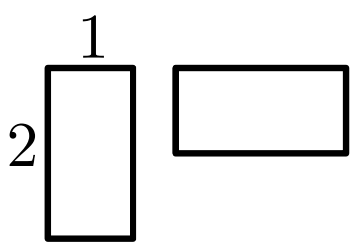
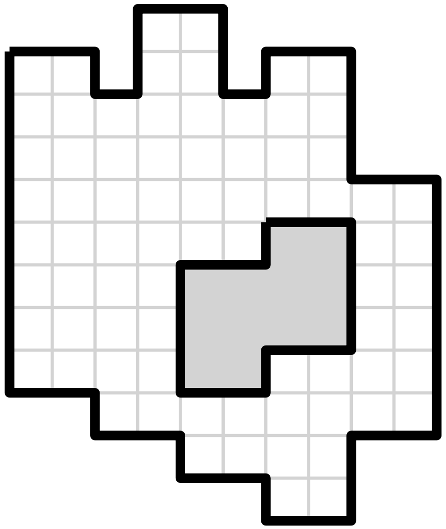
	No holes: $O(n)$ Holes: $O(n \log n)$	NP-complete
	$\widetilde{O}(n^3)$	$\widetilde{O}(n^3)$

Tiling with 2×2 squares



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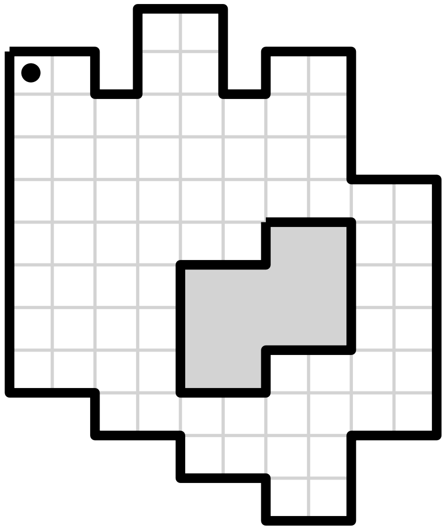

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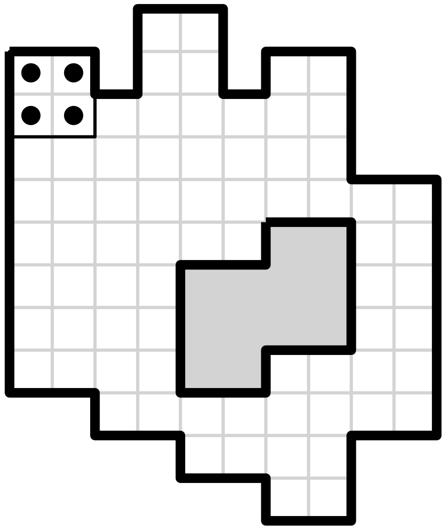

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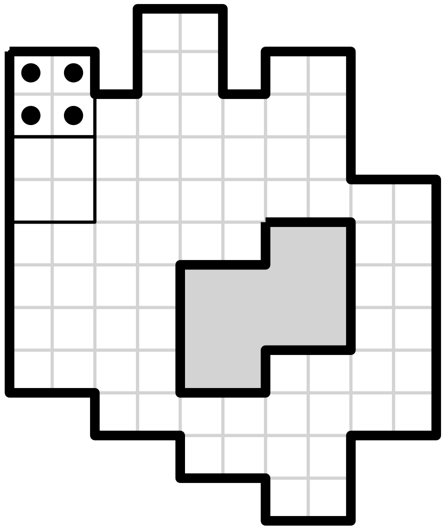

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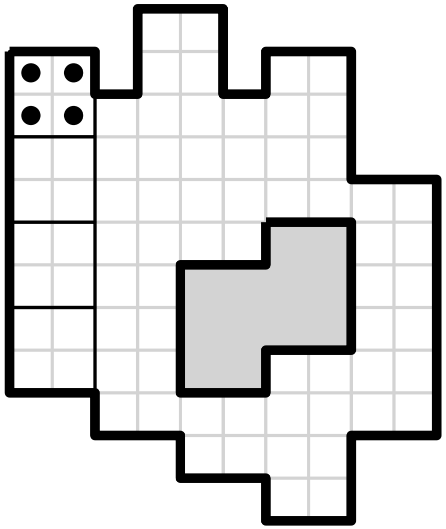

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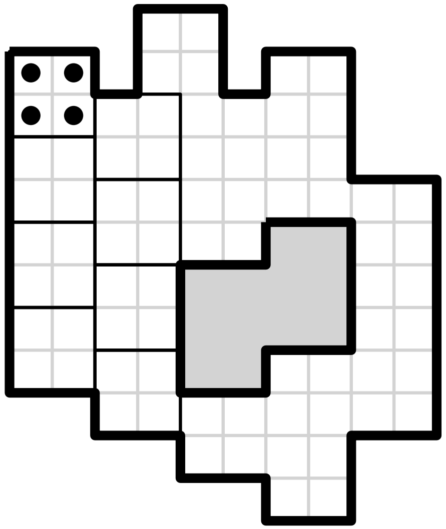

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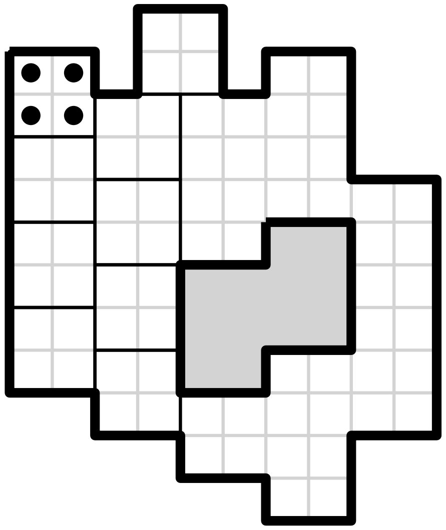

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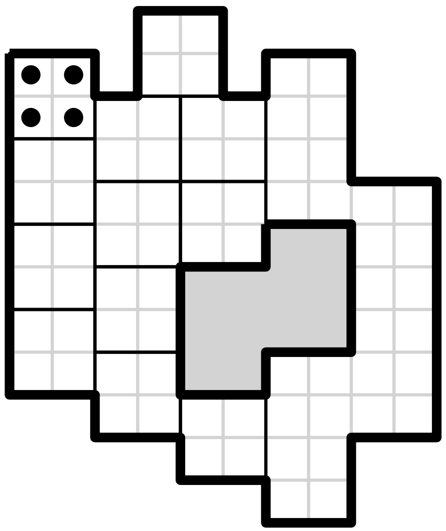

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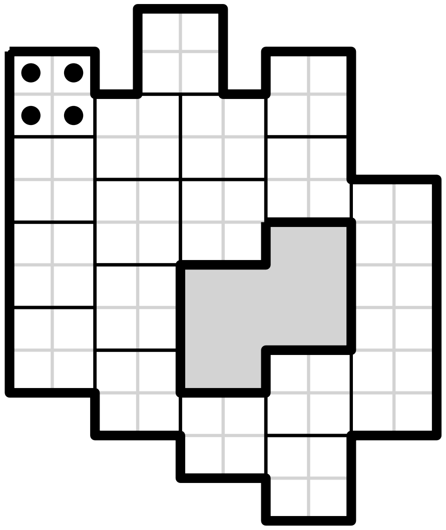

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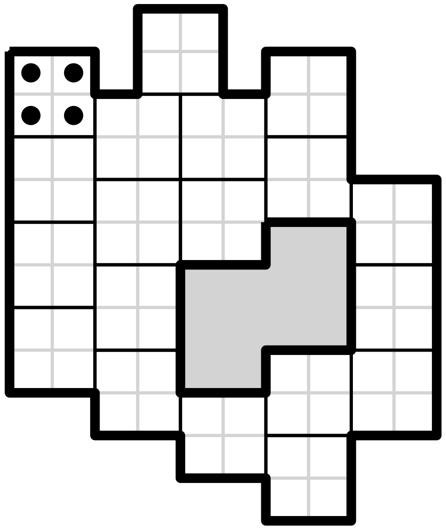

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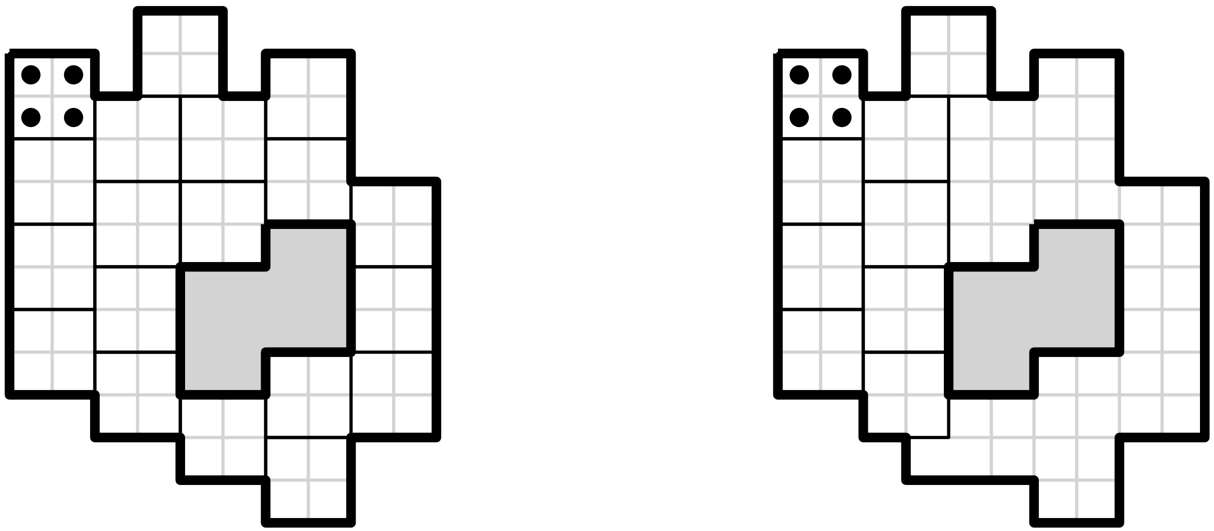

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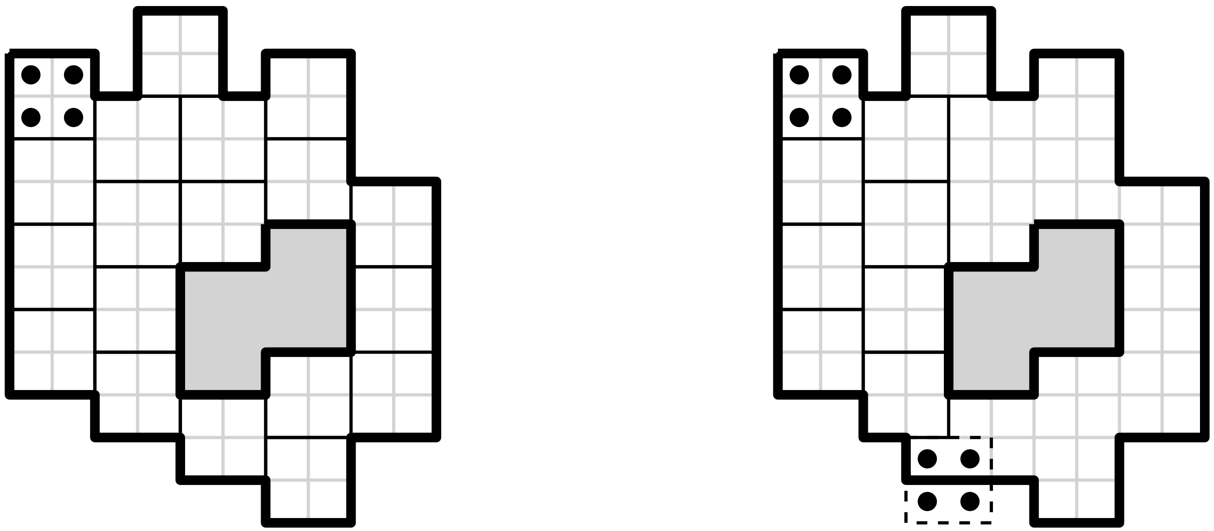

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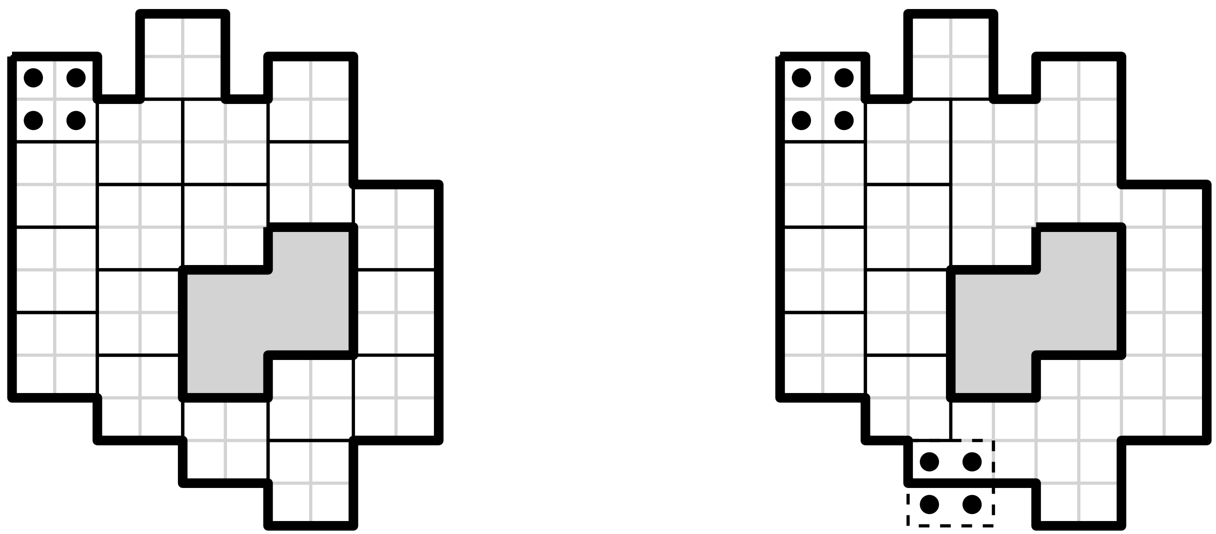
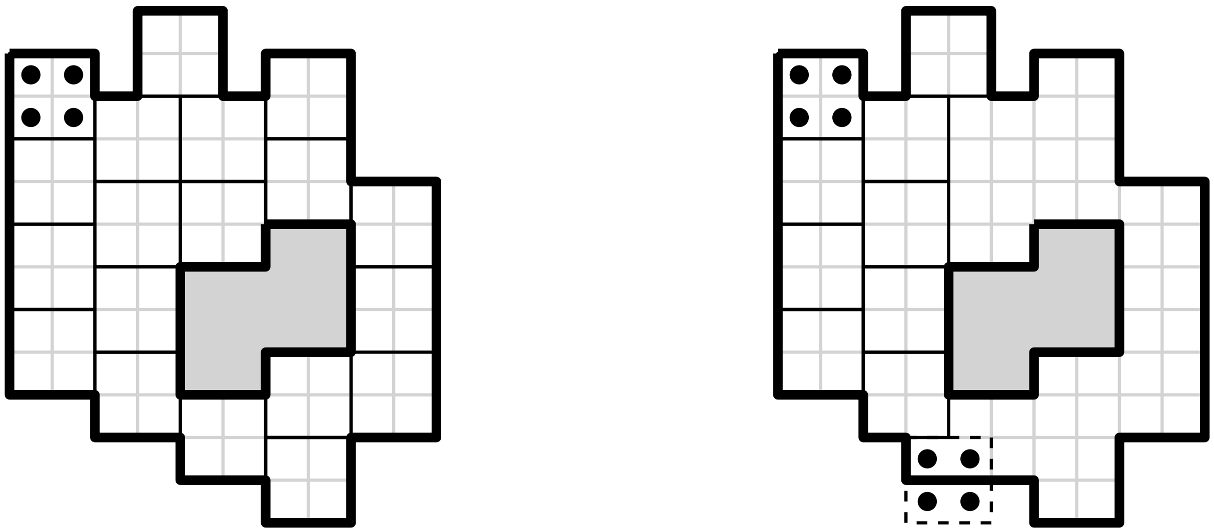


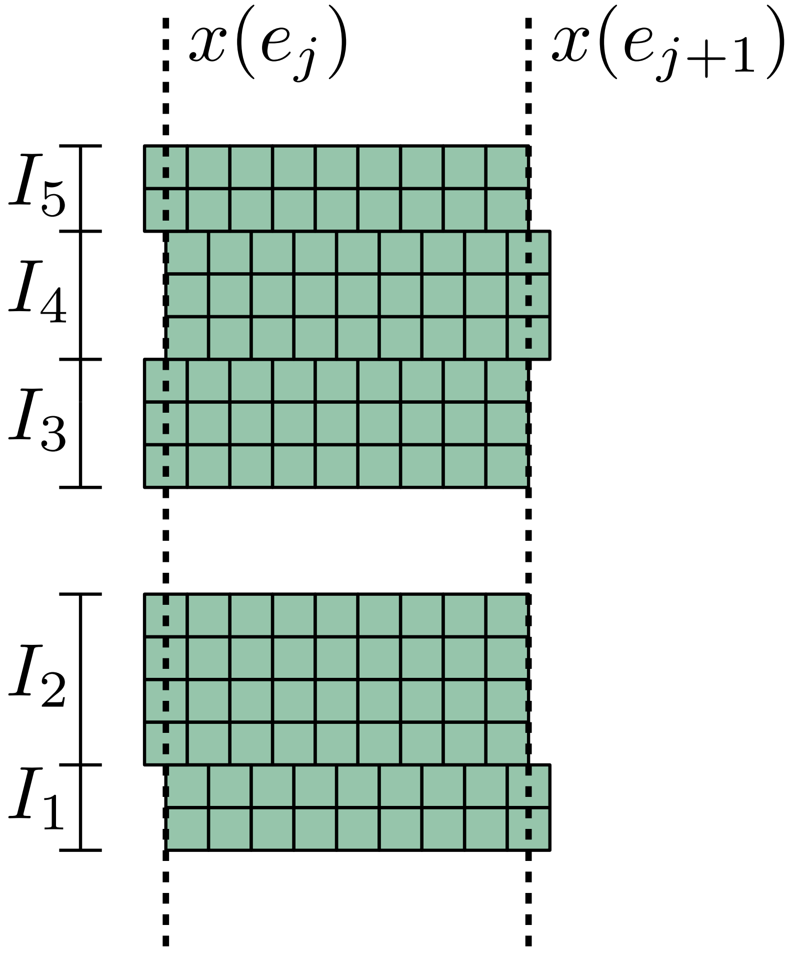
Can be done in $O(A)$ time.

Tiling with 2×2 squares

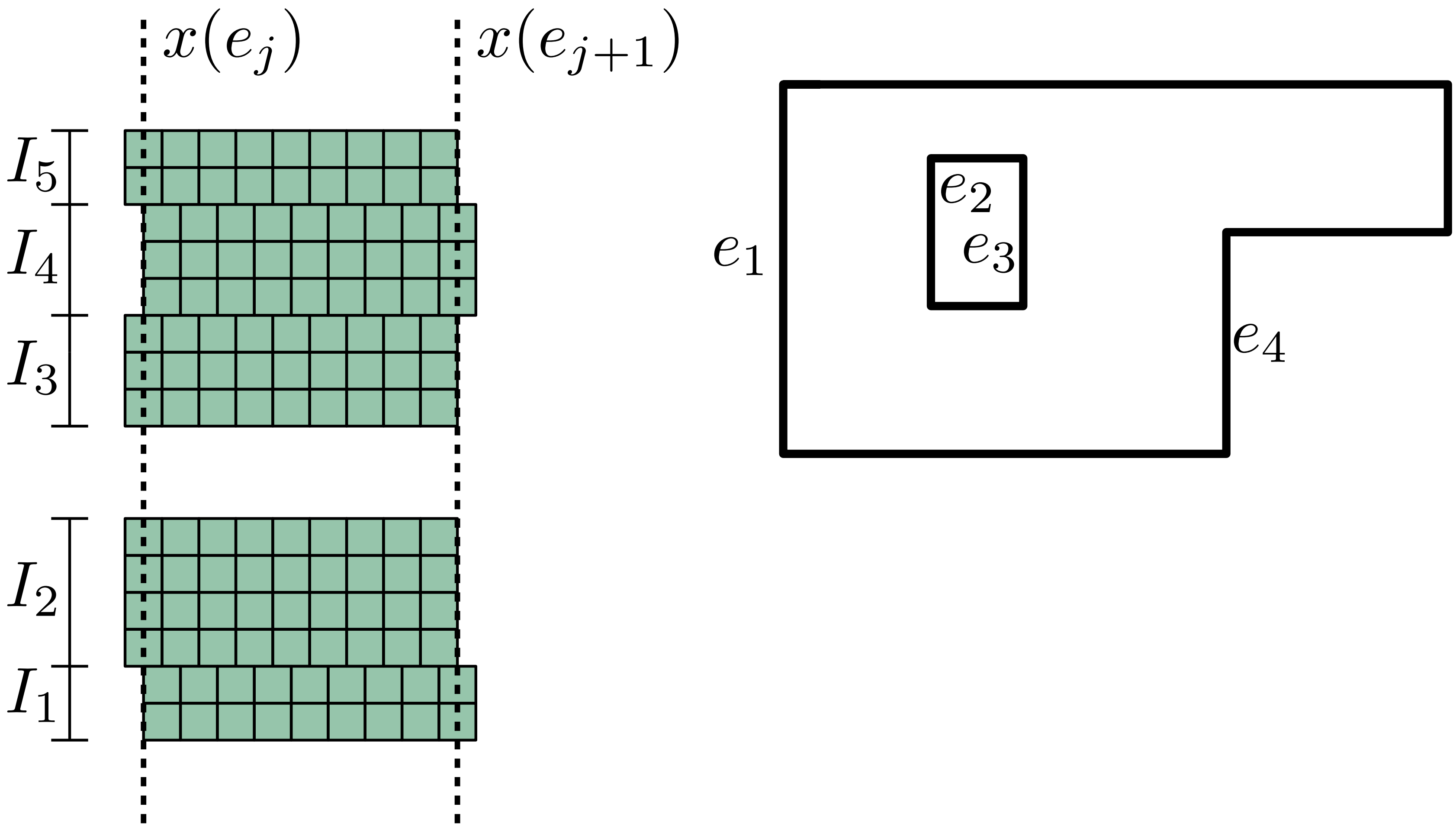


Can be done in $O(A)$ time.

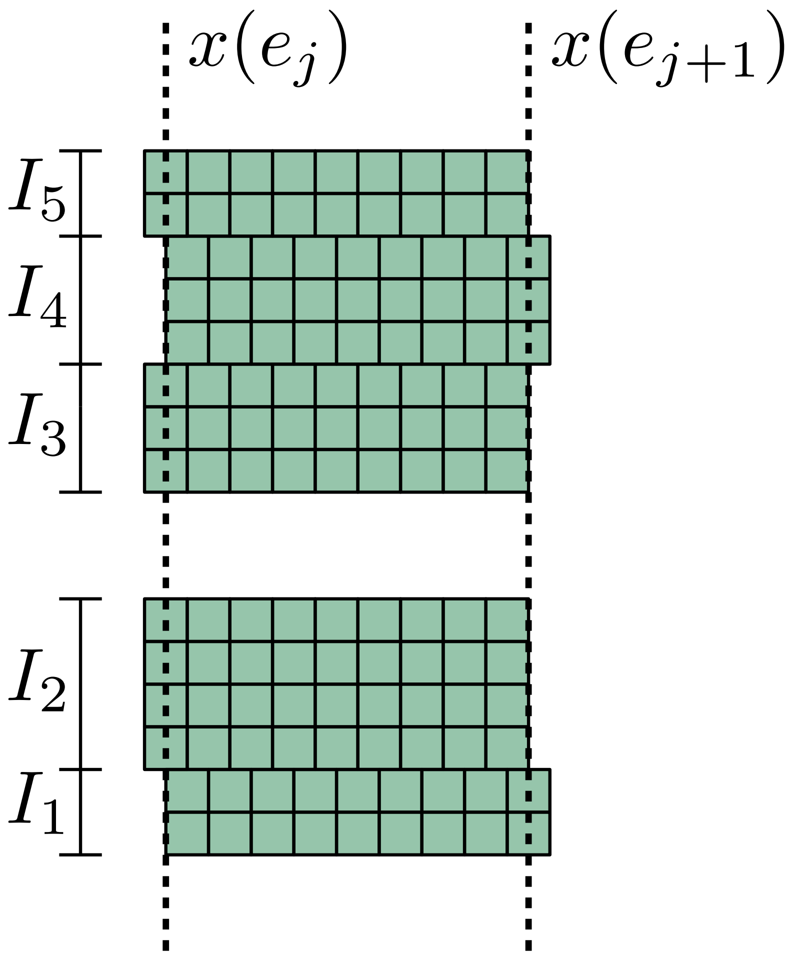
Polynomial-time algorithm but in the area of P!



$x(e_j)$
$x(e_{j+1})$
I_5
I_4
I_3
I_2
I_1



$x(e_j)$
$x(e_{j+1})$
I_5
I_4
I_3
I_2
I_1
e_1
e_2
e_3
e_4


$x(e_j)$
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I_5
I_4
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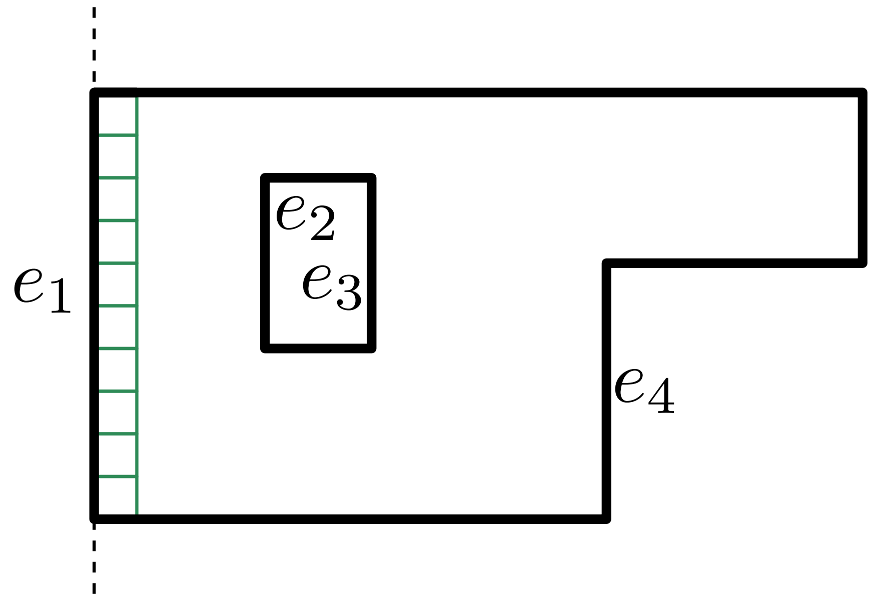
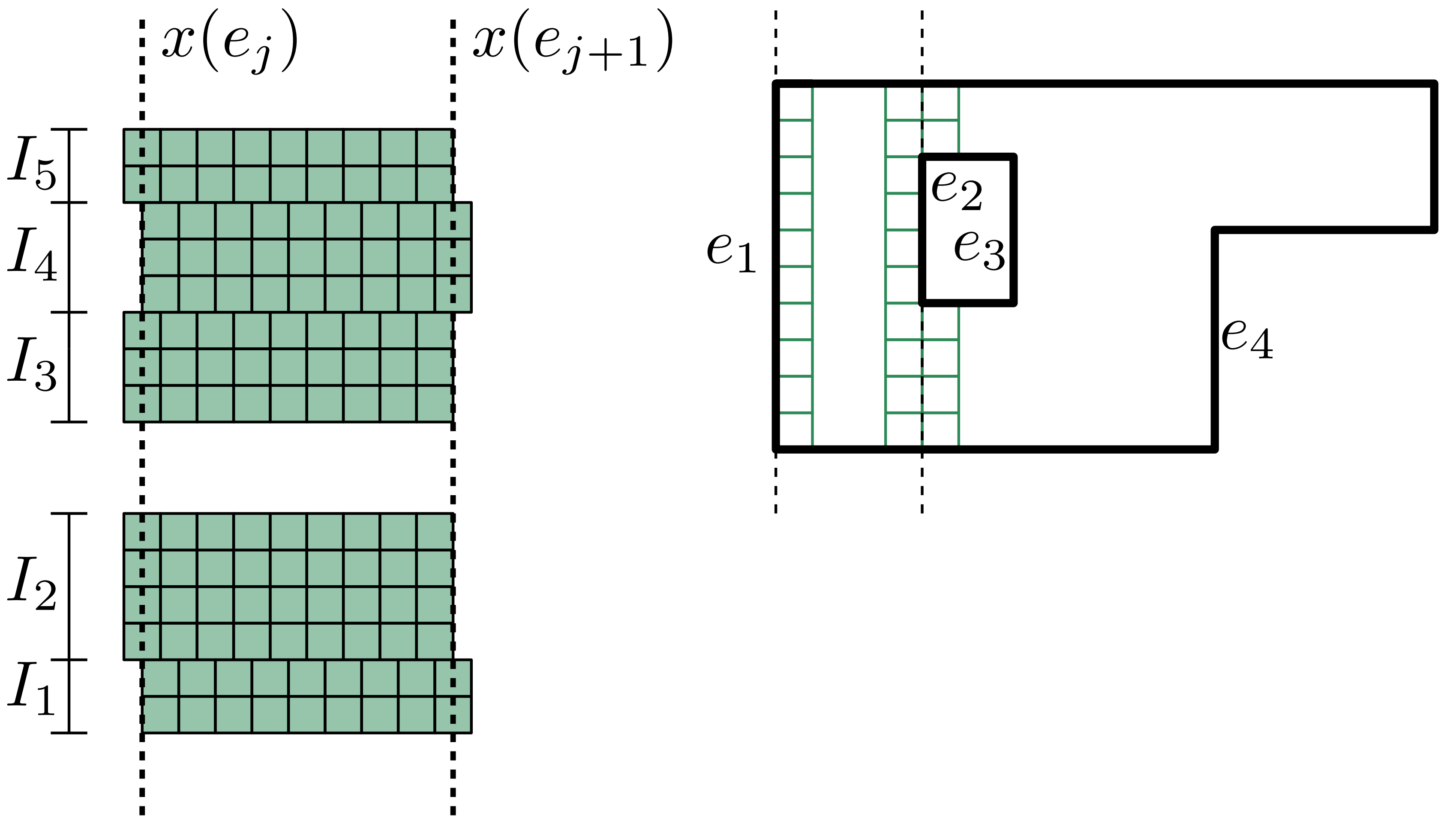

e_1
e_2
e_3
e_4


$x(e_j)$
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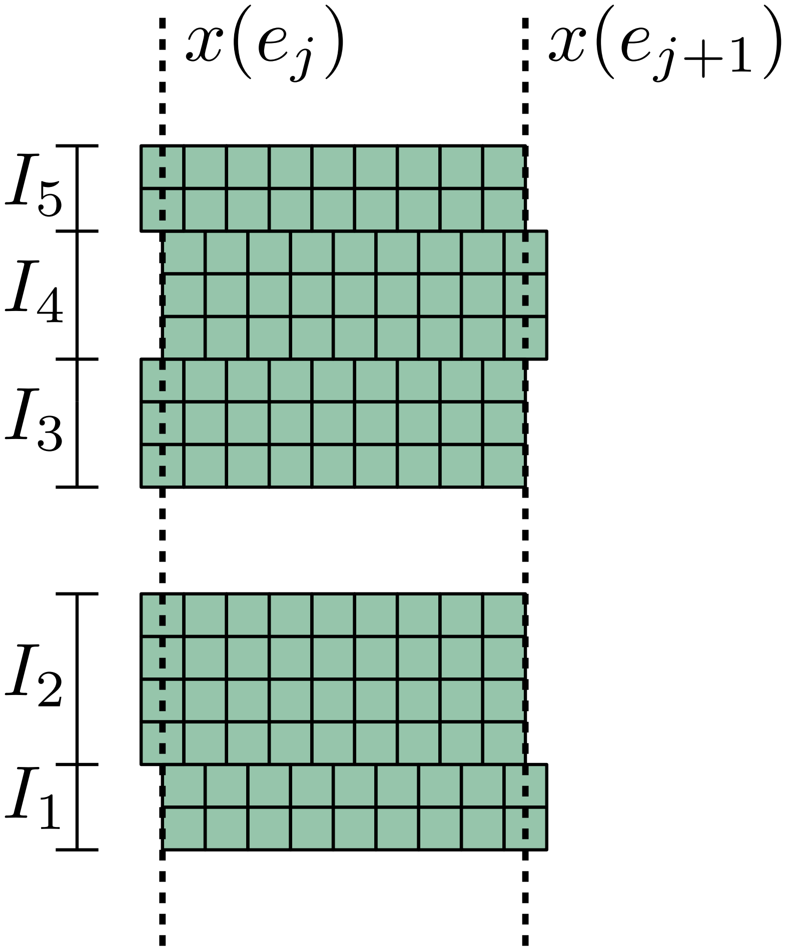
I_4
I_3
I_2
I_1
e_1
e_2
e_3
e_4


$x(e_j)$
$x(e_{j+1})$
I_5
I_4
I_3
I_2
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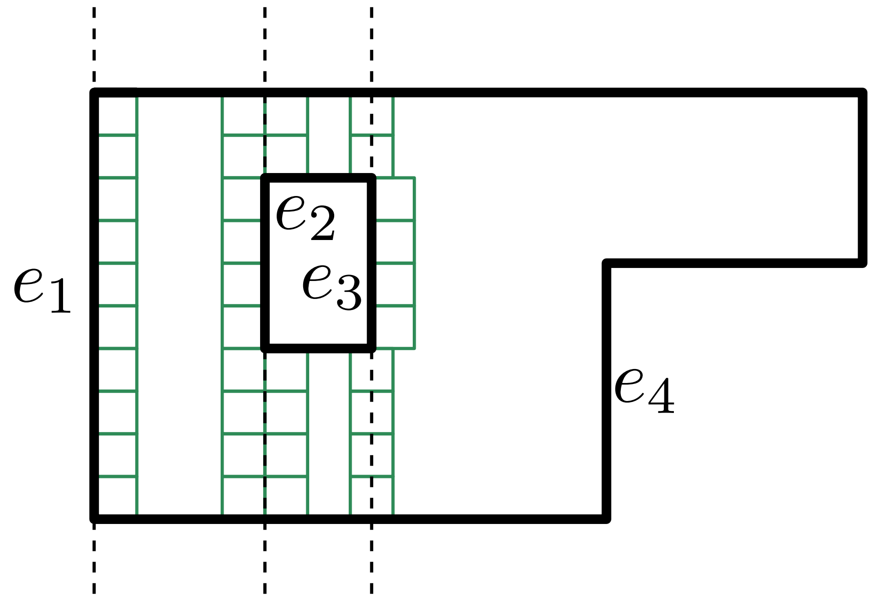
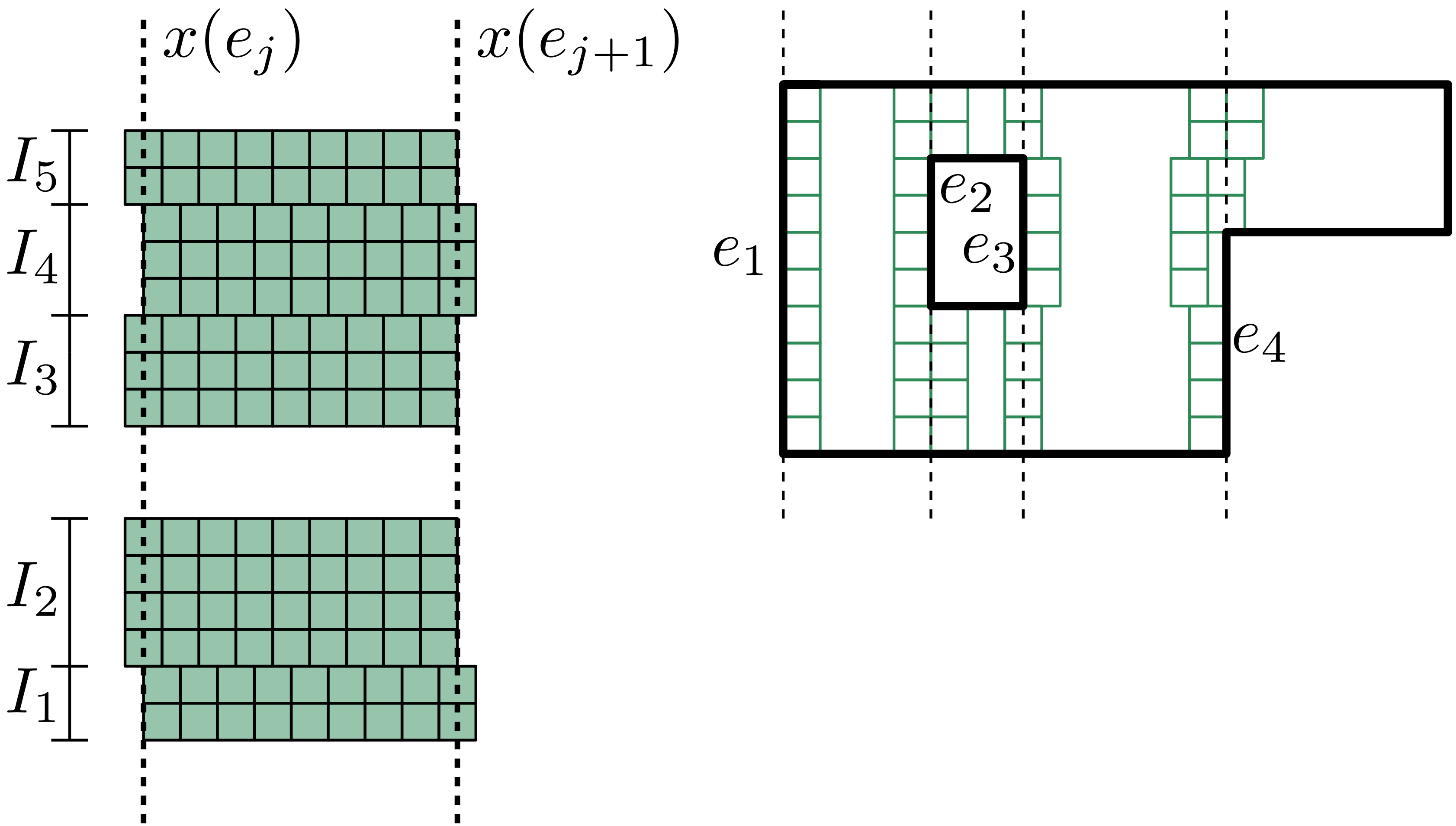

e_1
e_2
e_3
e_4



$x(e_j)$
$x(e_{j+1})$
I_5
I_4
I_3
I_2
I_1
e_1
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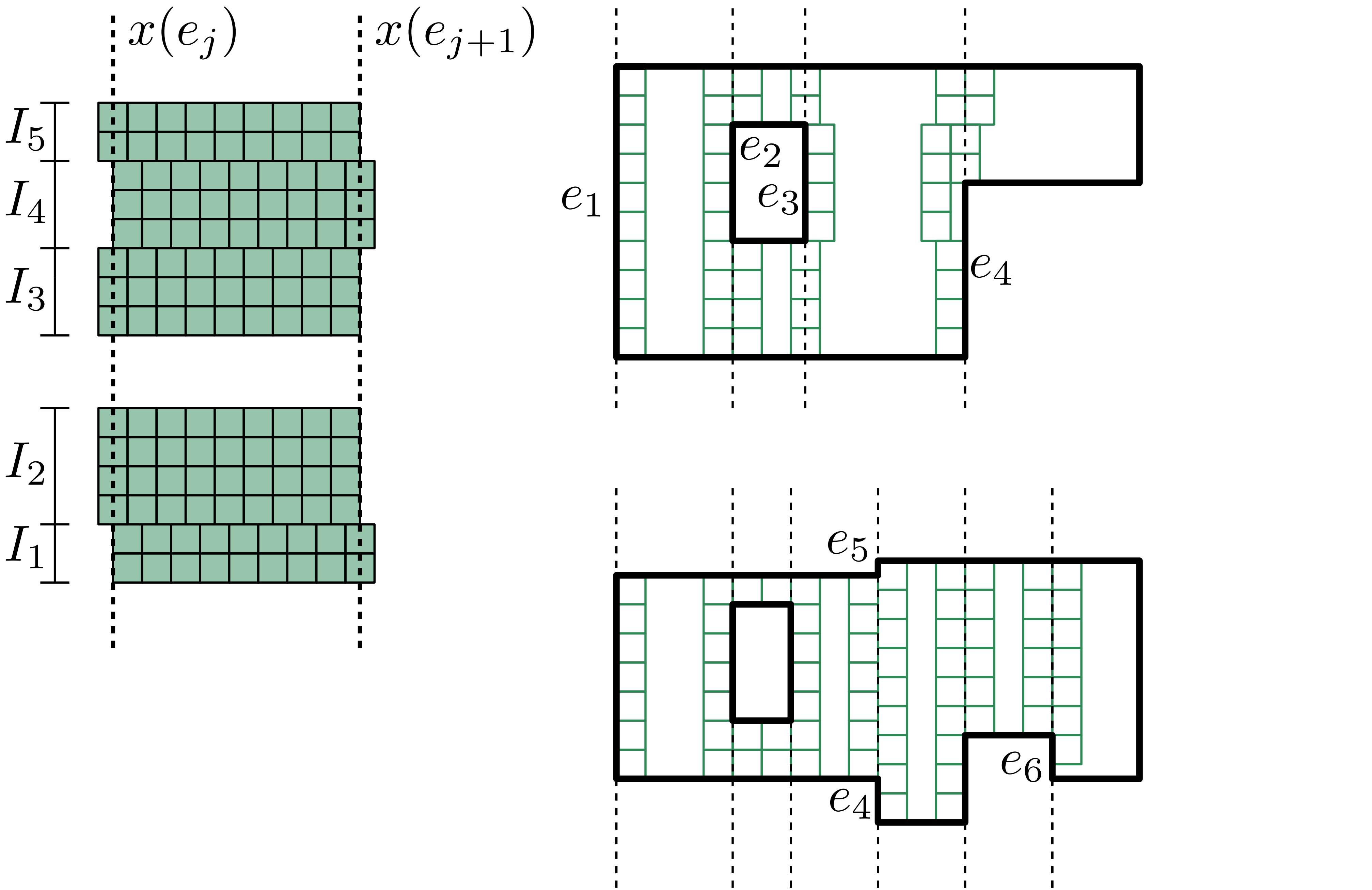
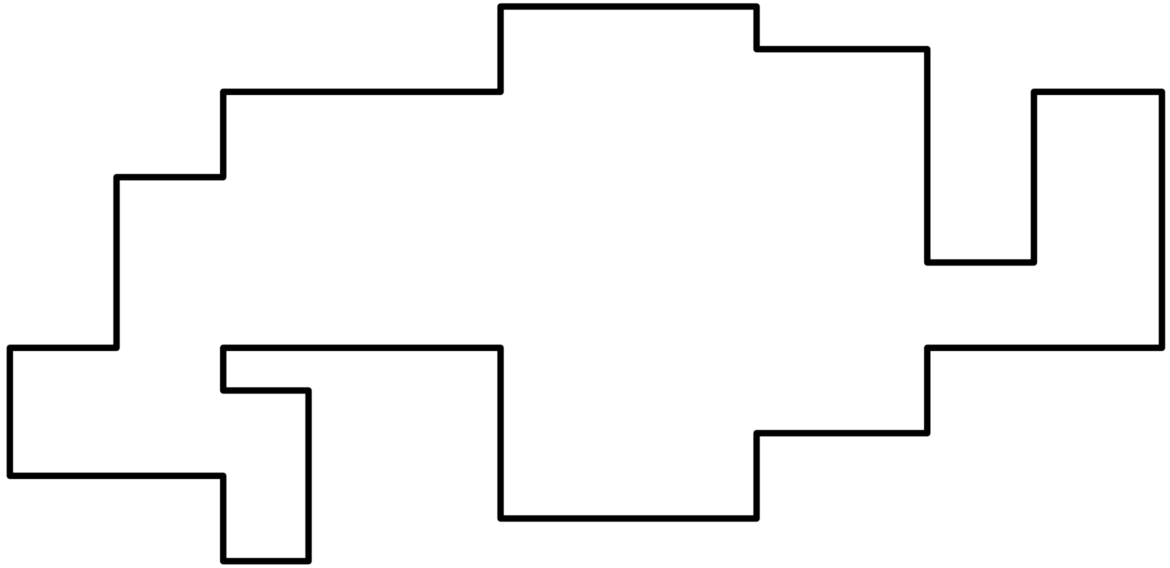
e_3
e_4



$x(e_j)$
$x(e_{j+1})$
I_5
I_4
I_3
I_2
I_1
e_1
e_2
e_3
e_4
e_5
e_4
e_6

No holes: $O(n)$ time!



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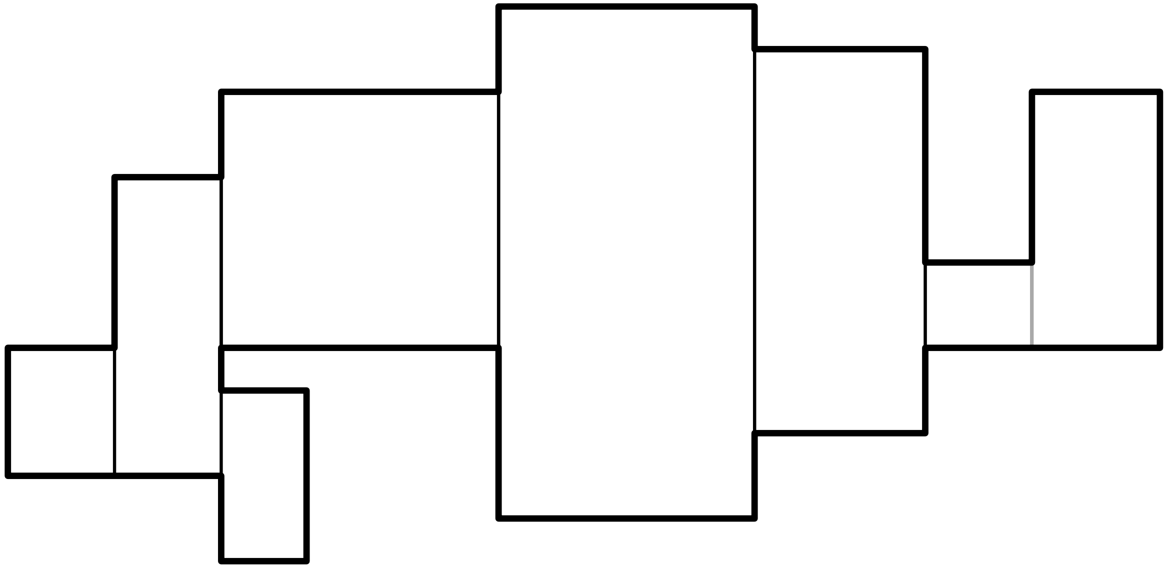

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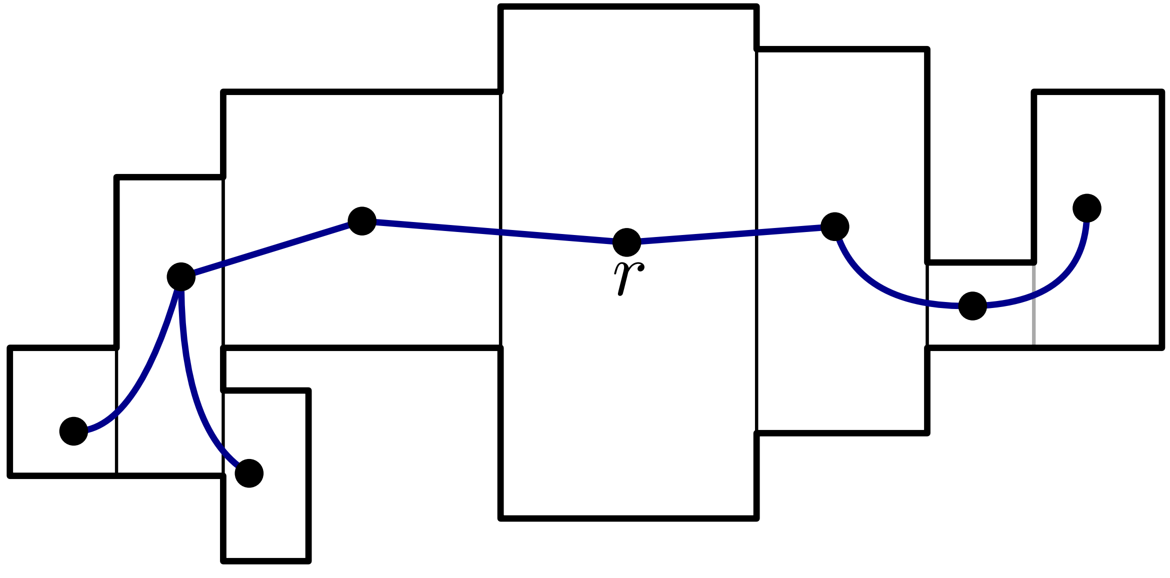

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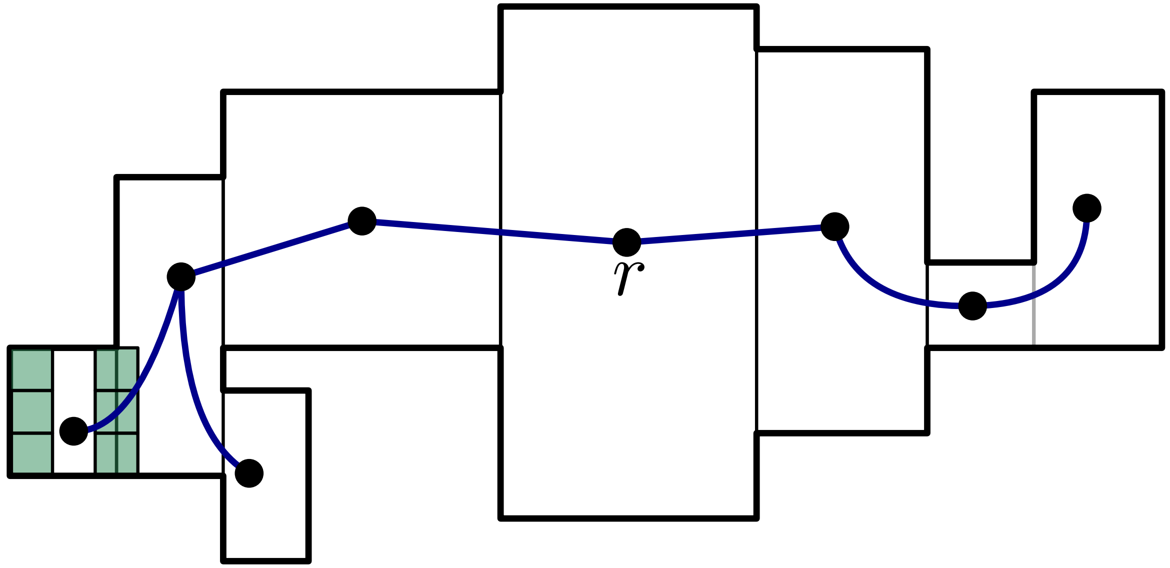

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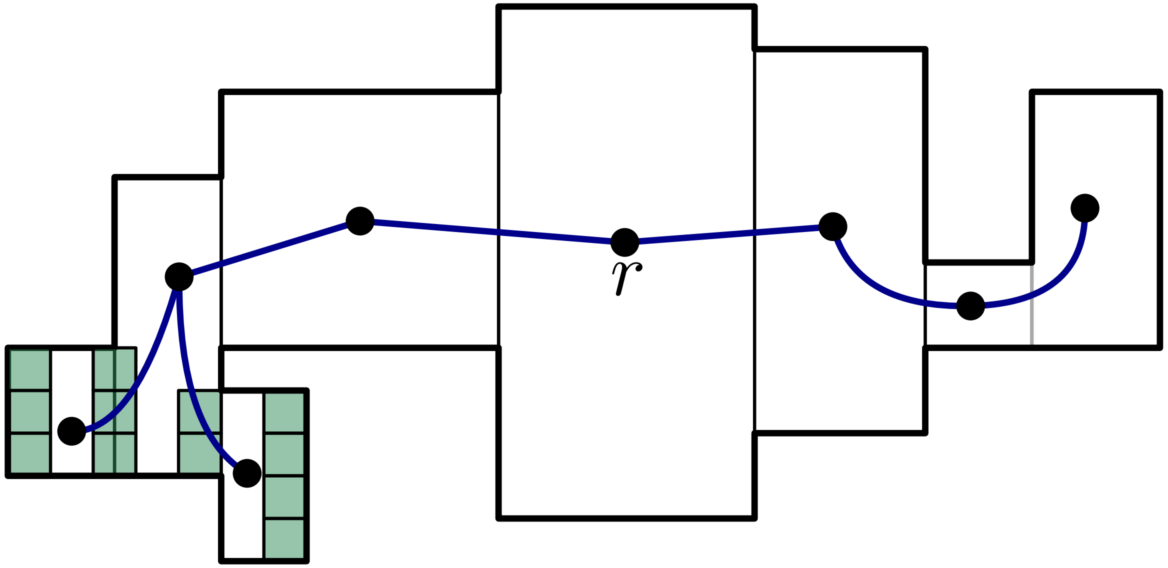

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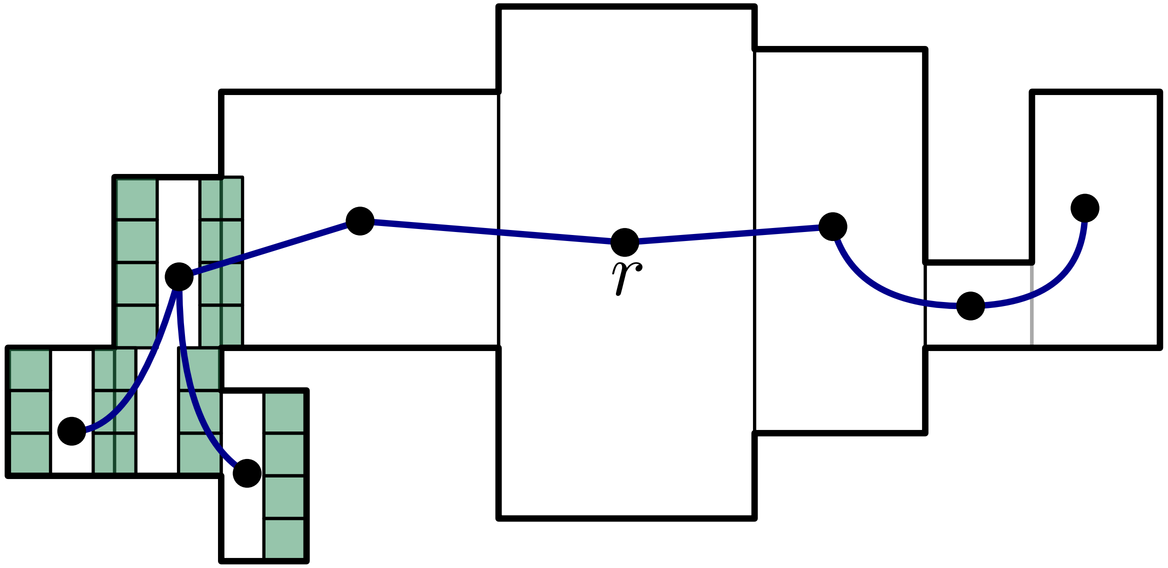

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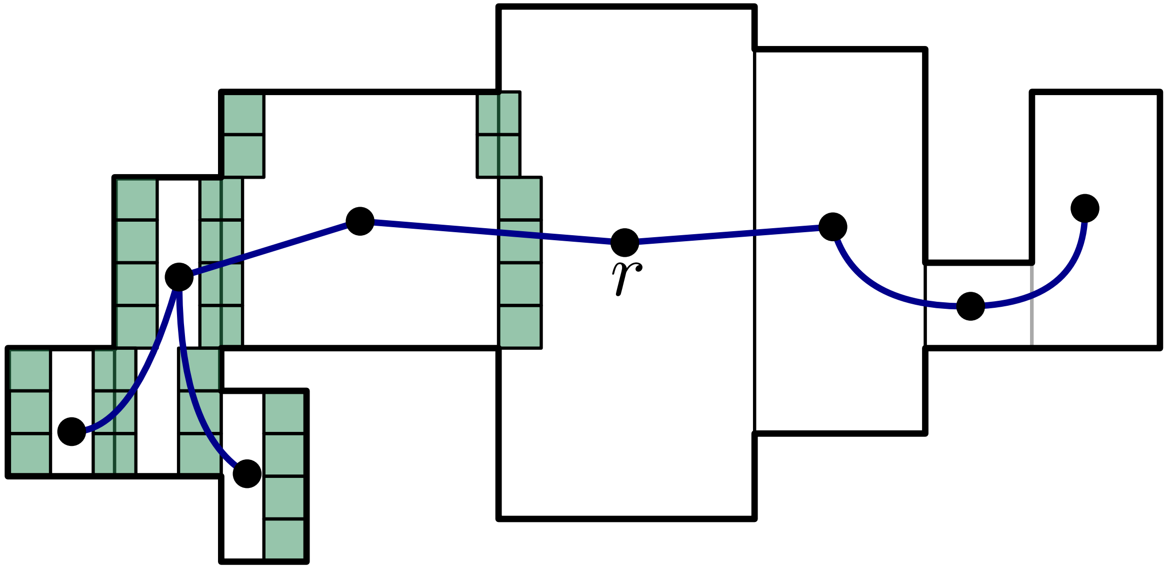

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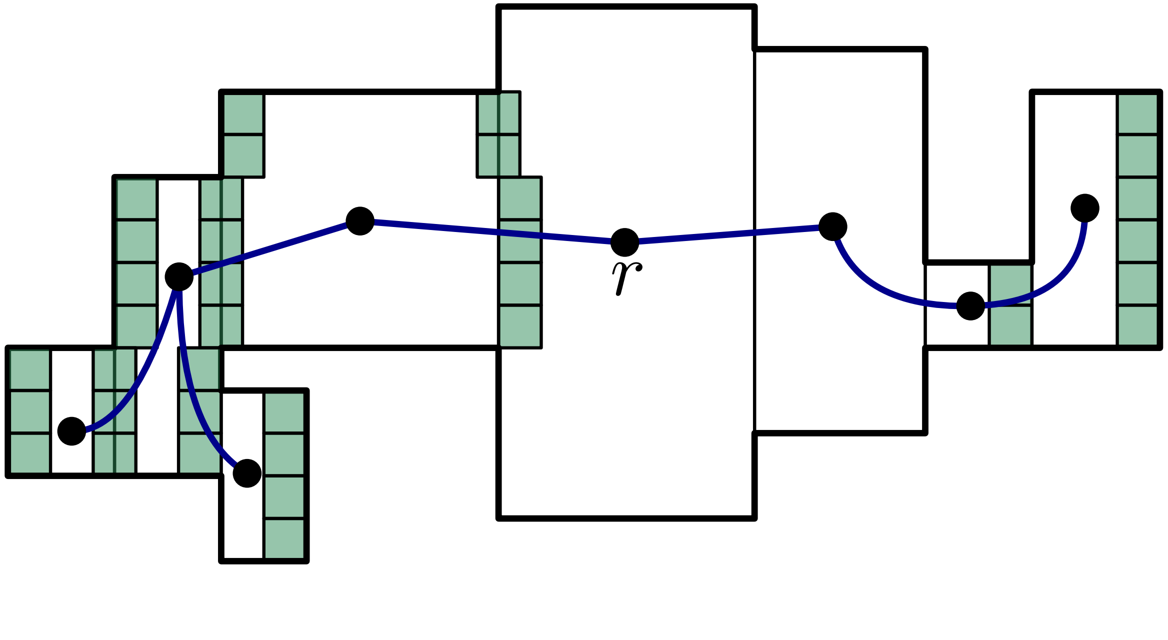

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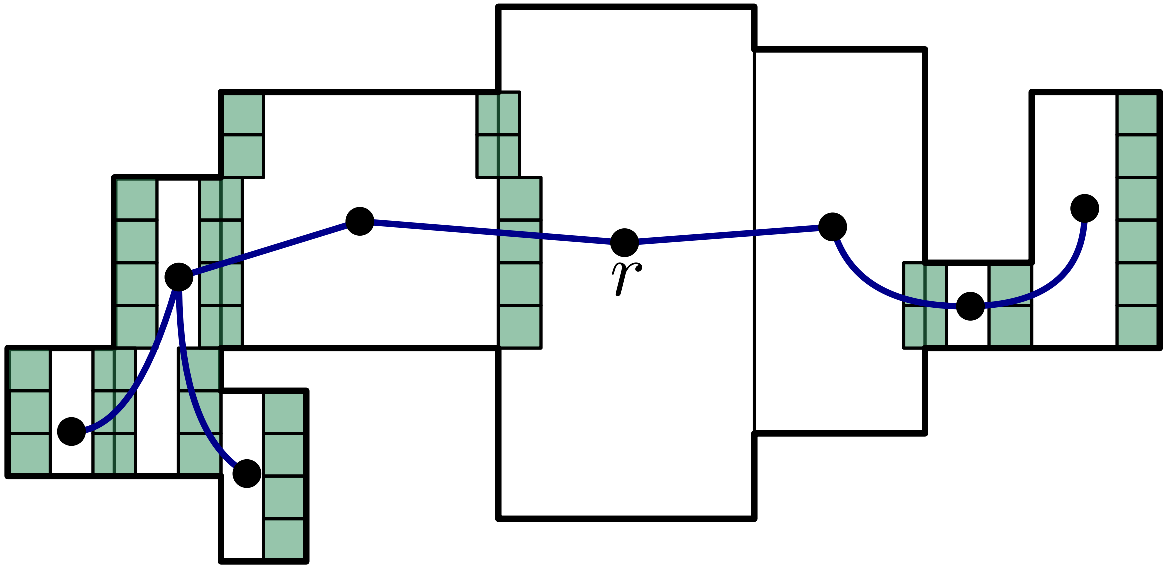

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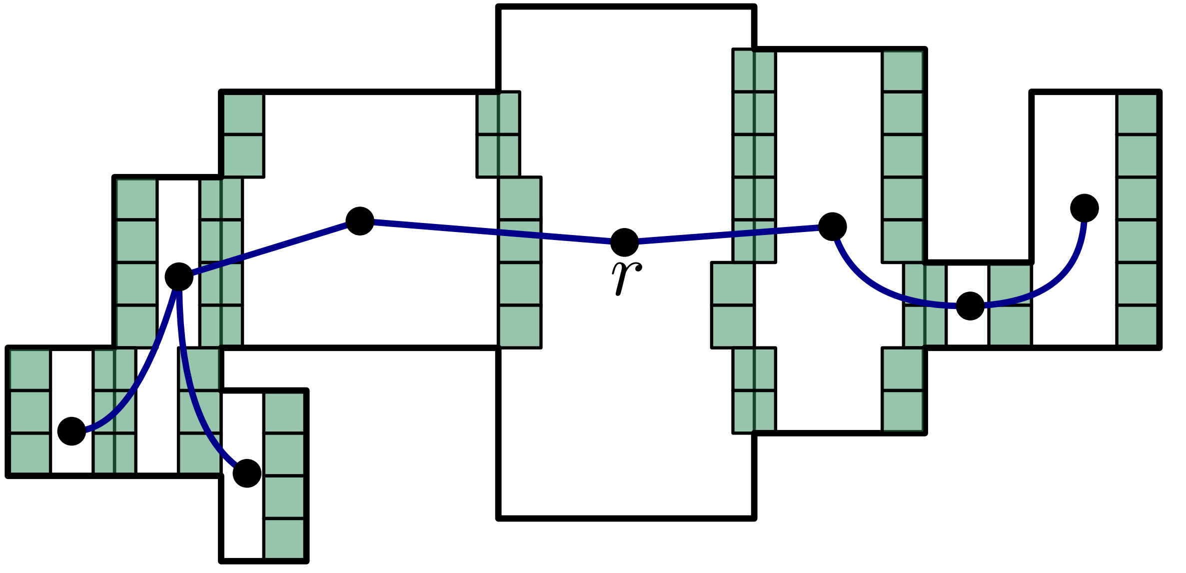

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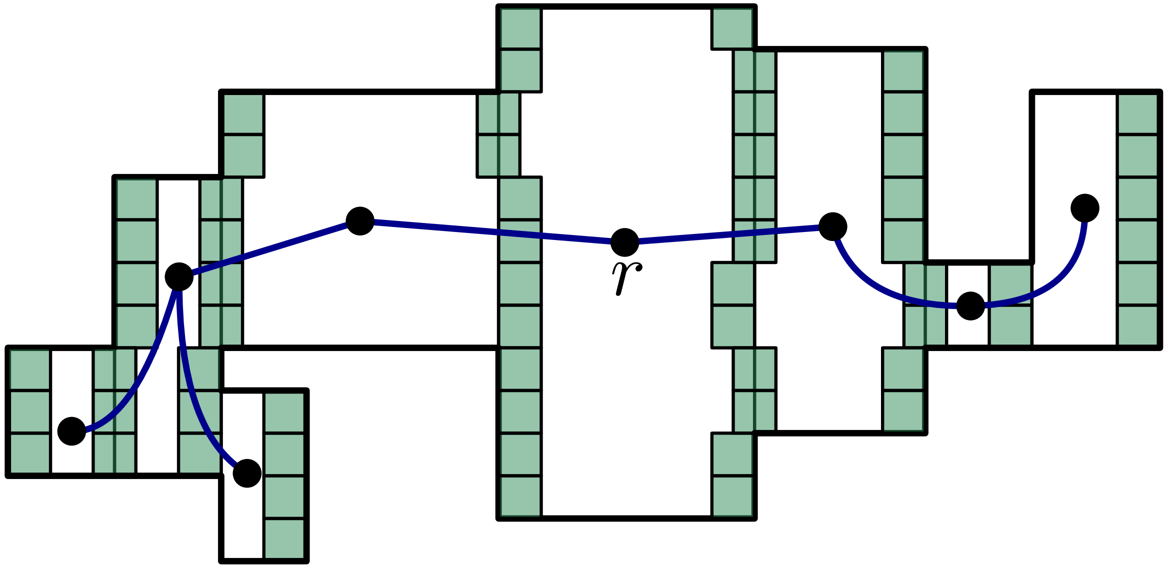

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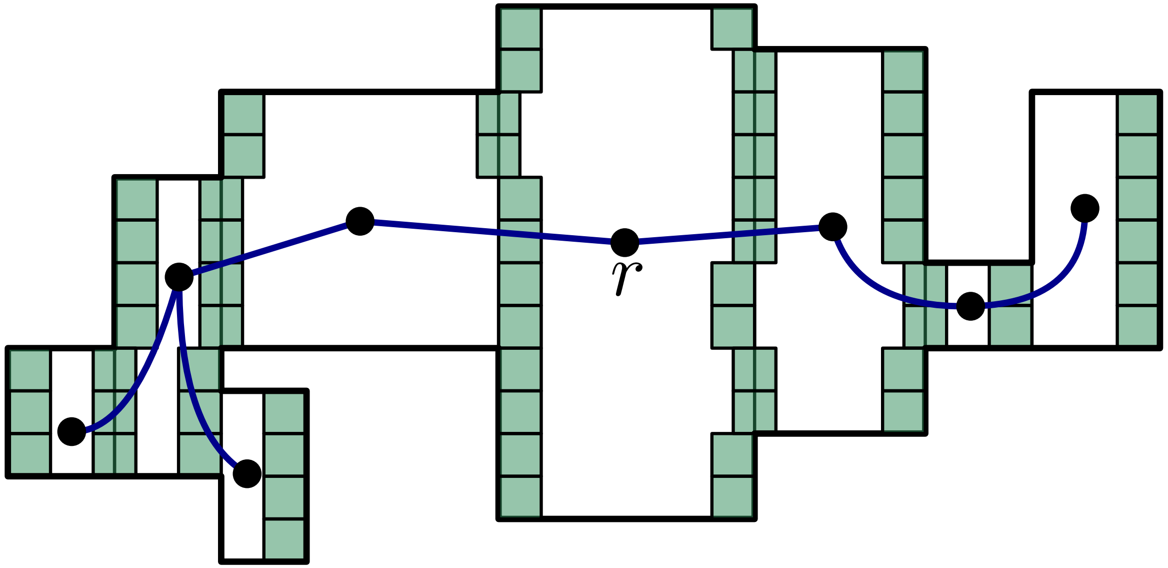

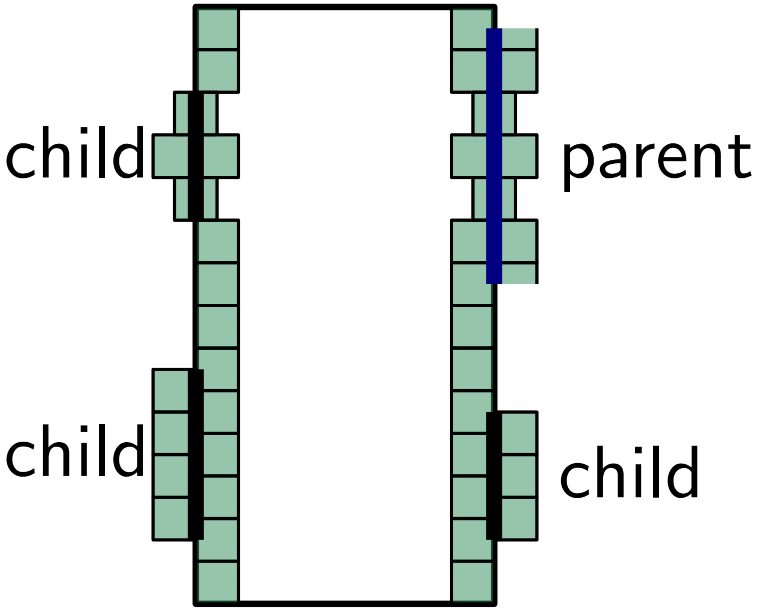

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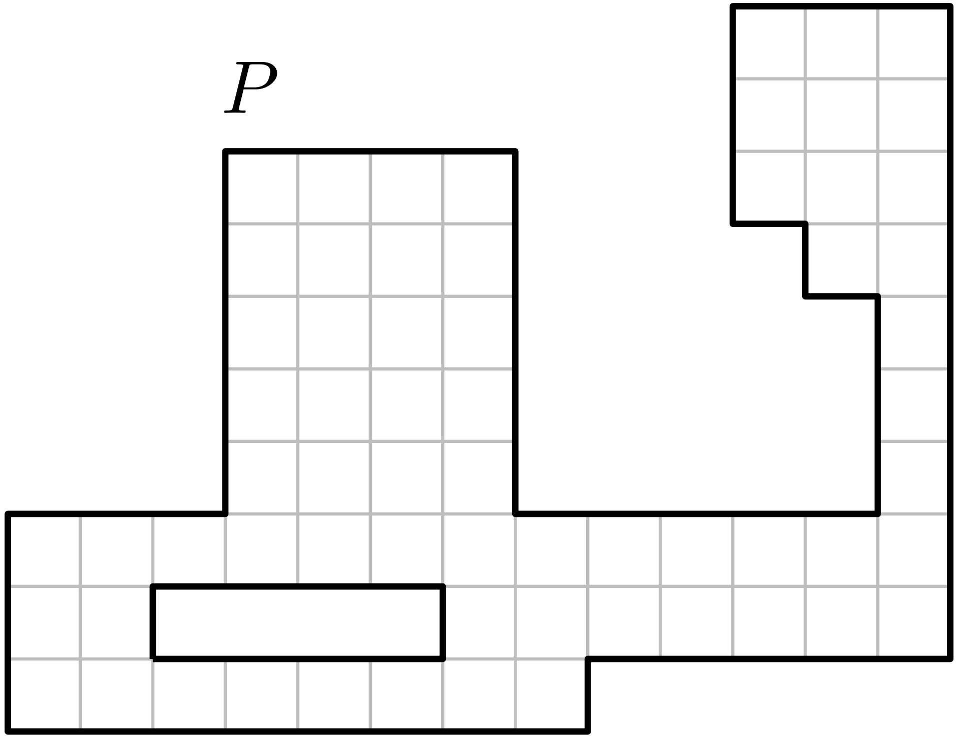

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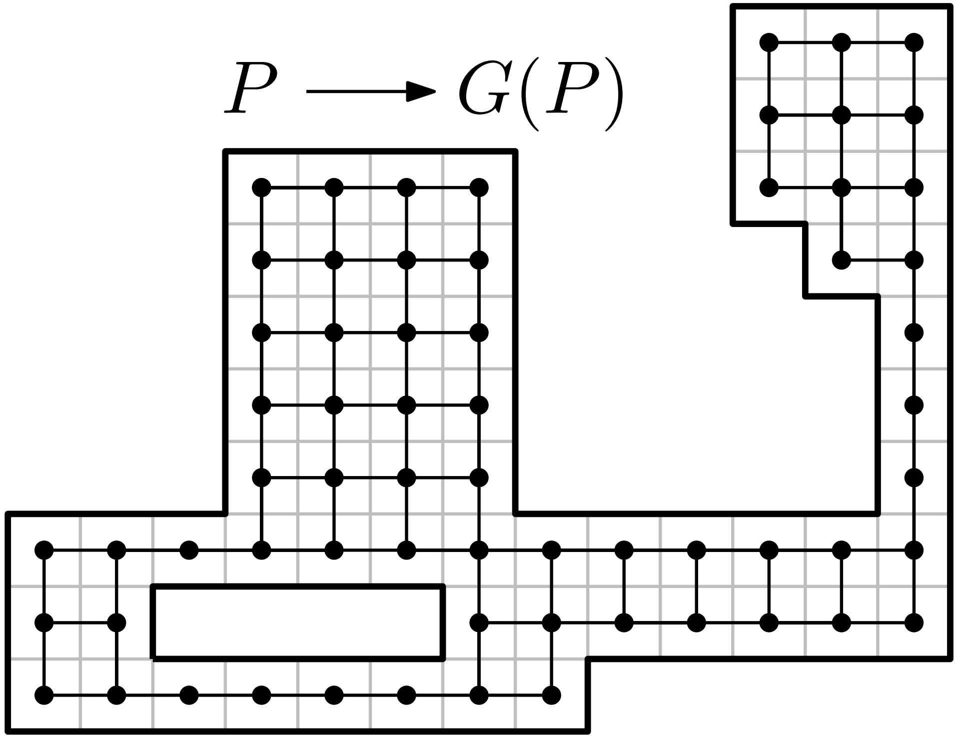

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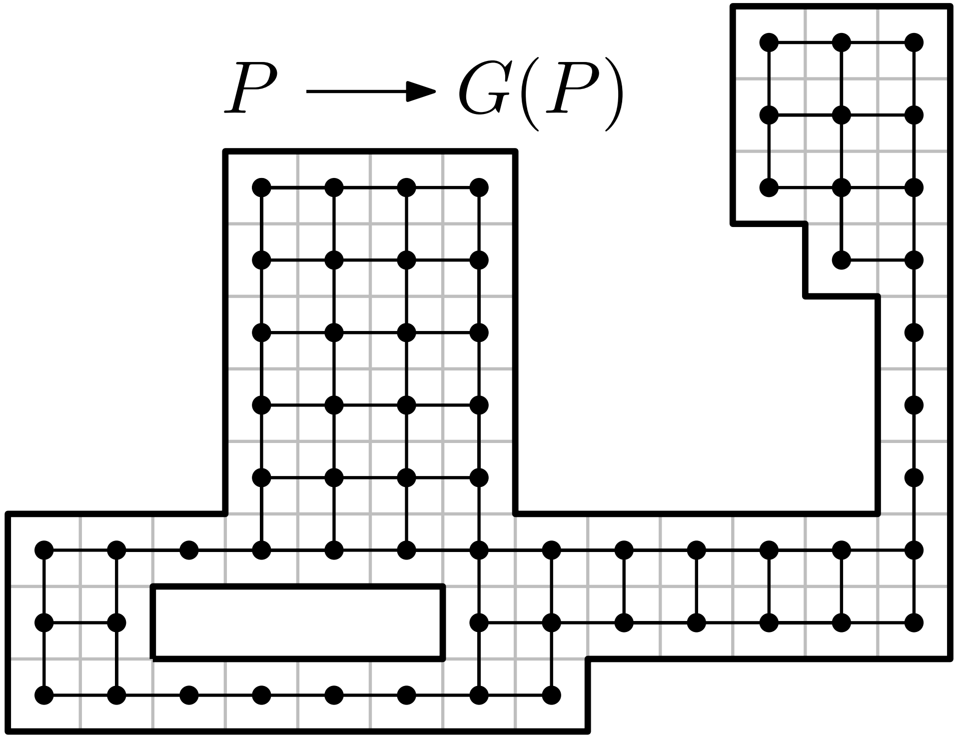

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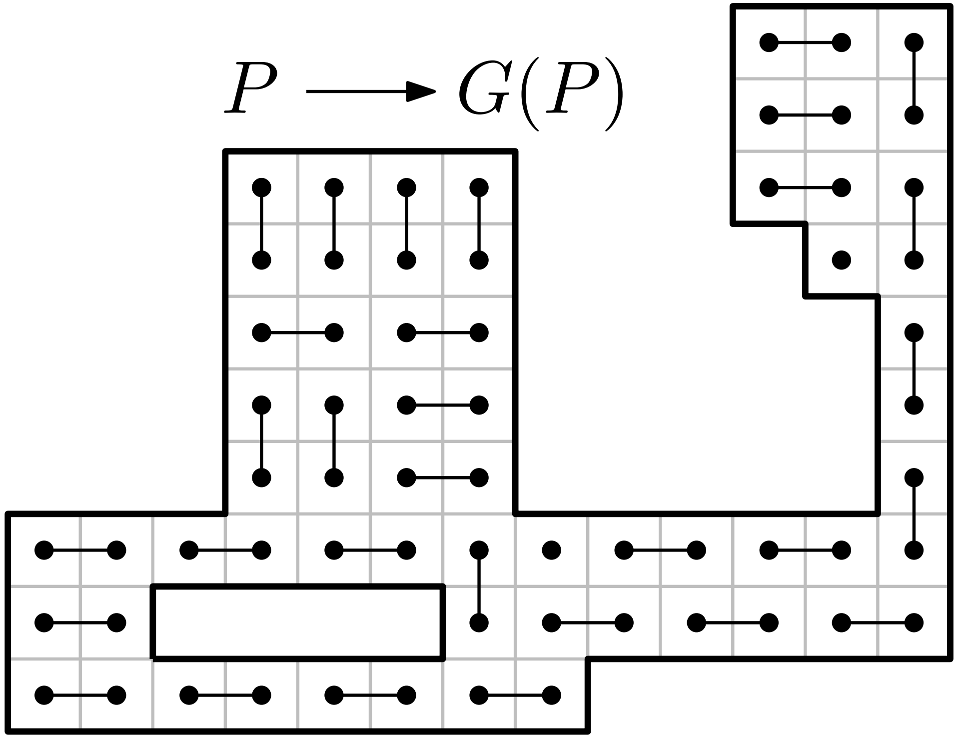

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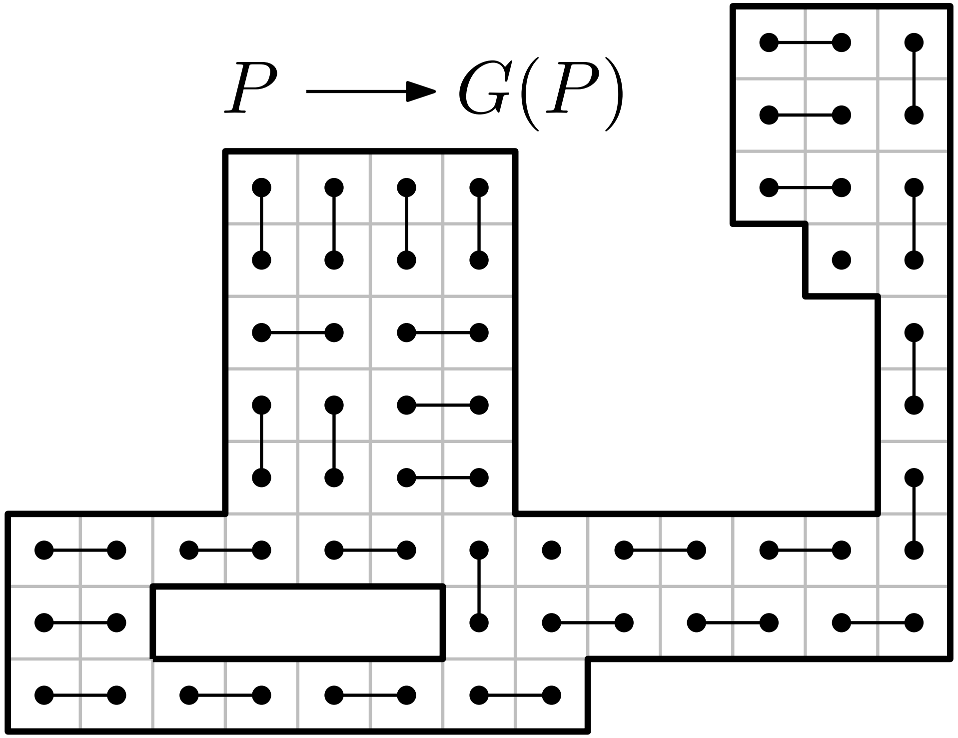

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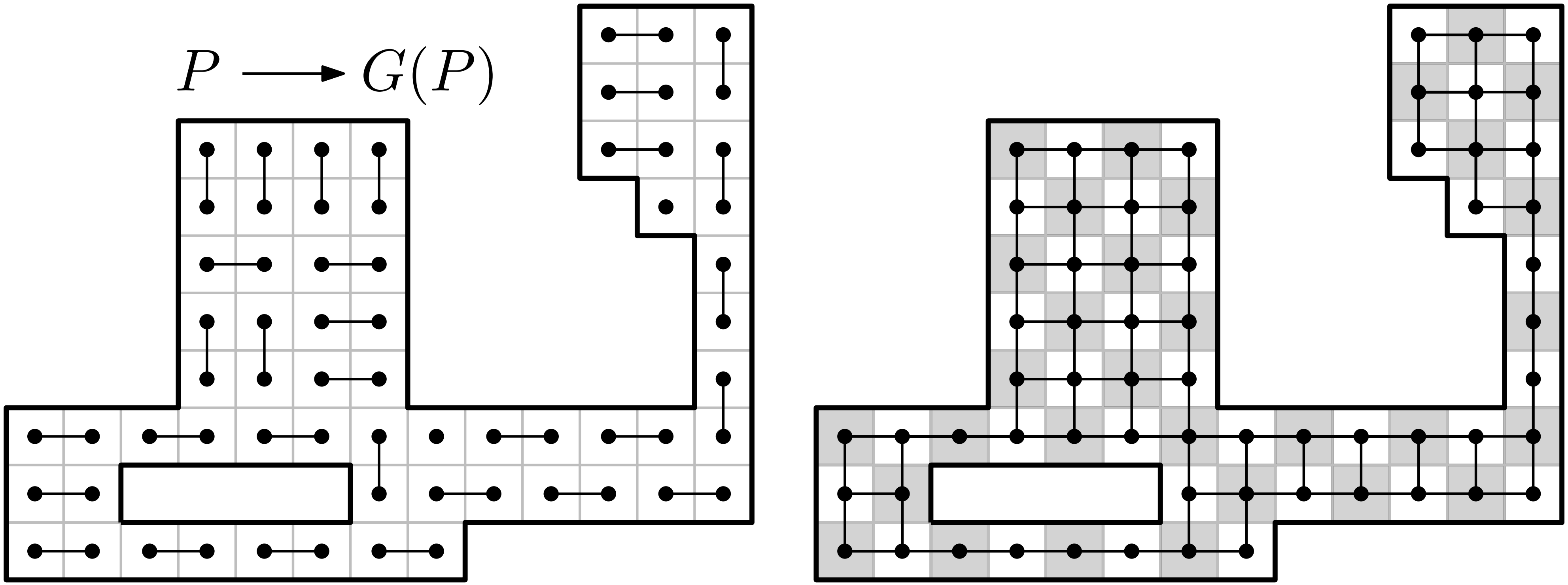

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Multiple source multiple sink maximum flow: $\widetilde{O}(A)$ [Borradaile et al., SICOMP 2017].

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This Talk

Packing Dominos in $\widetilde{O}(n^3)$ time

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Assume no holes

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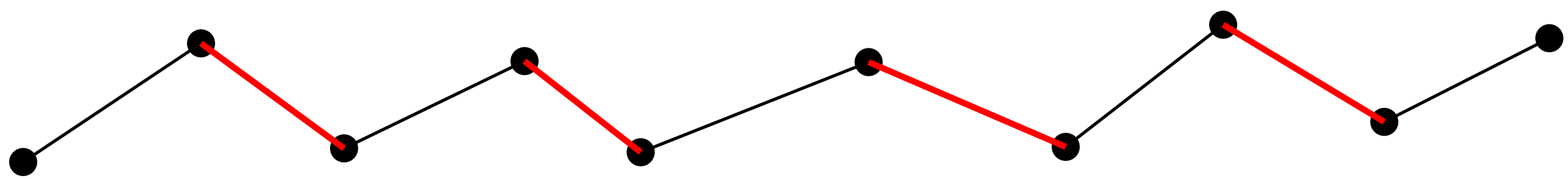

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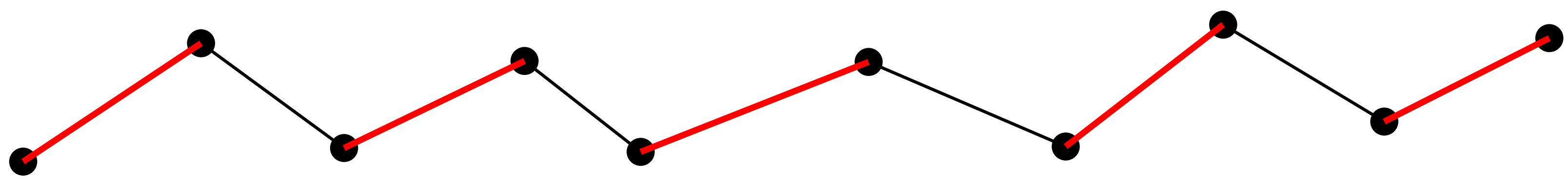

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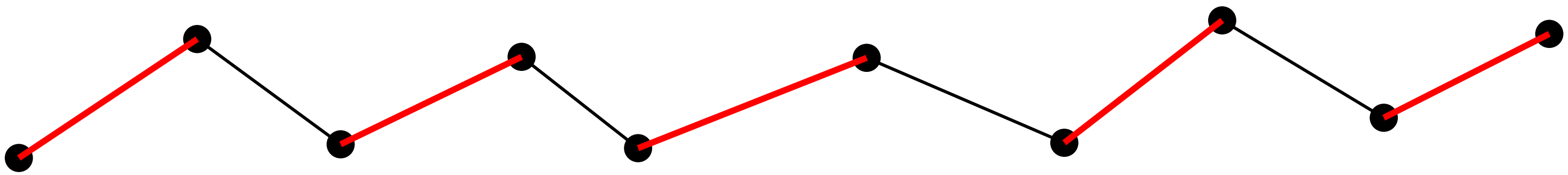

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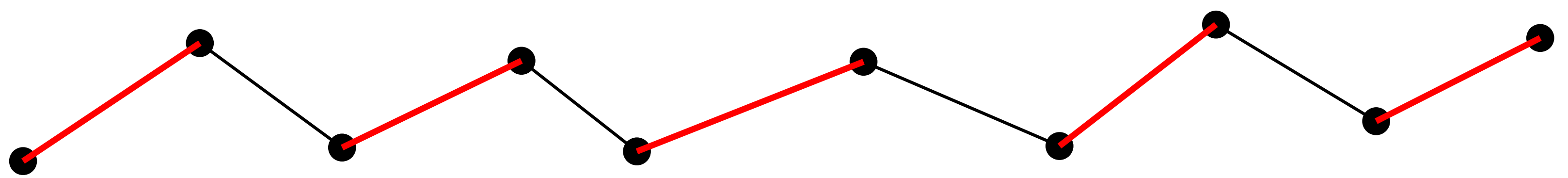

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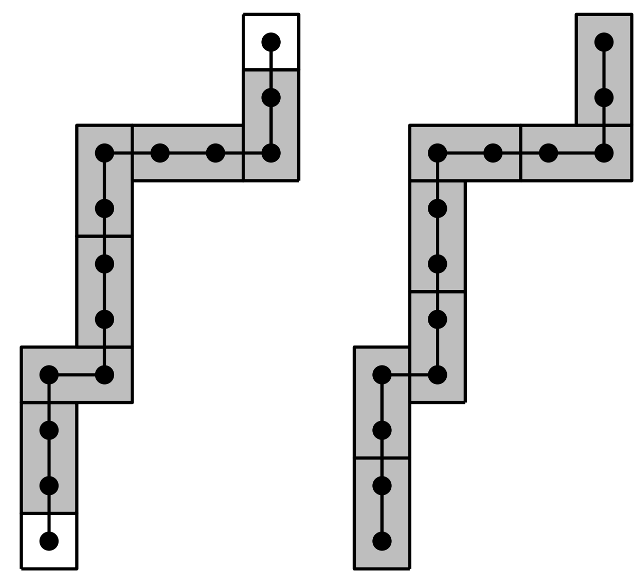

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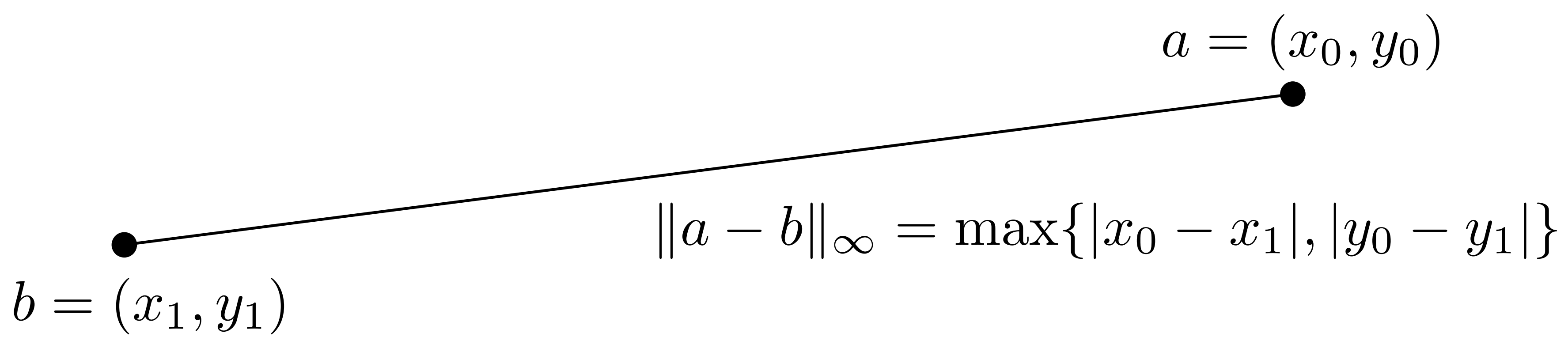

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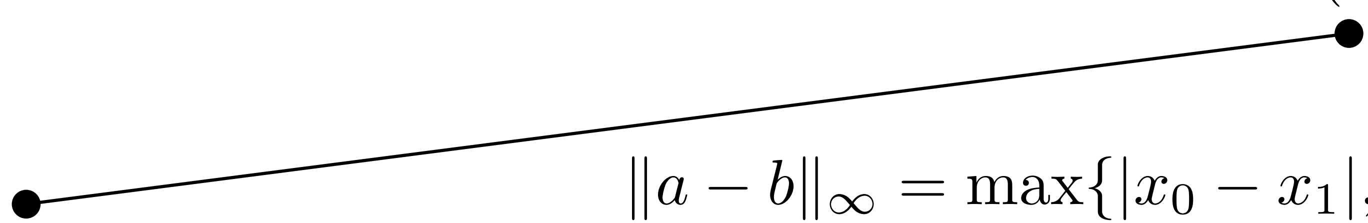

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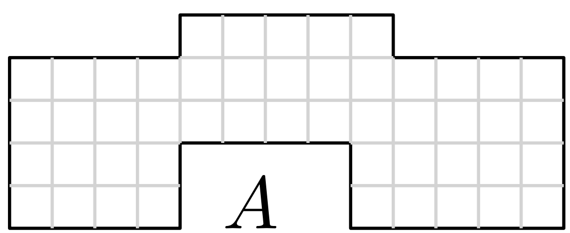

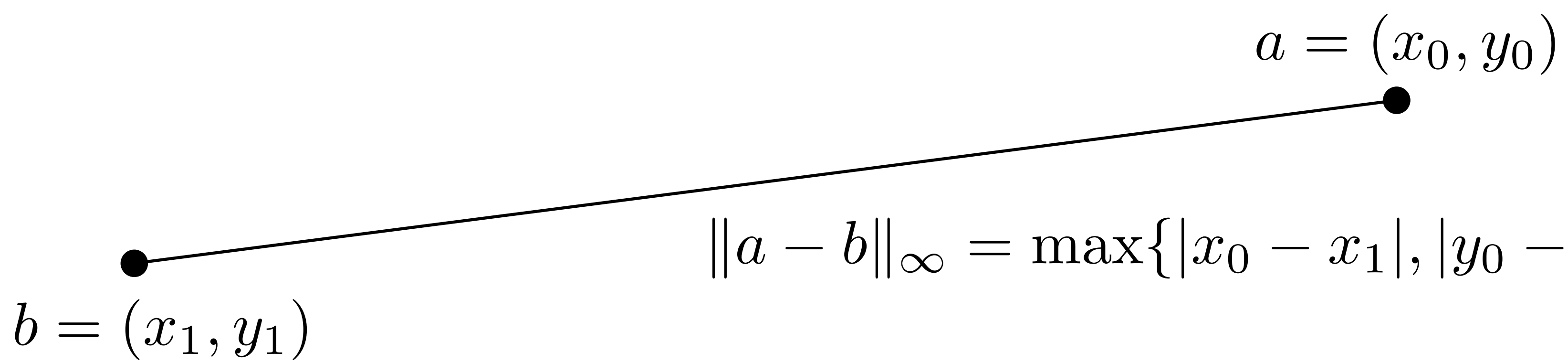

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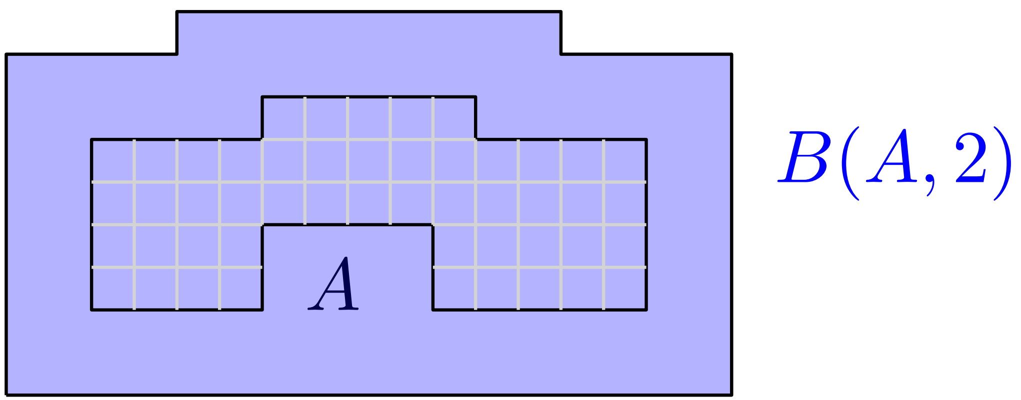

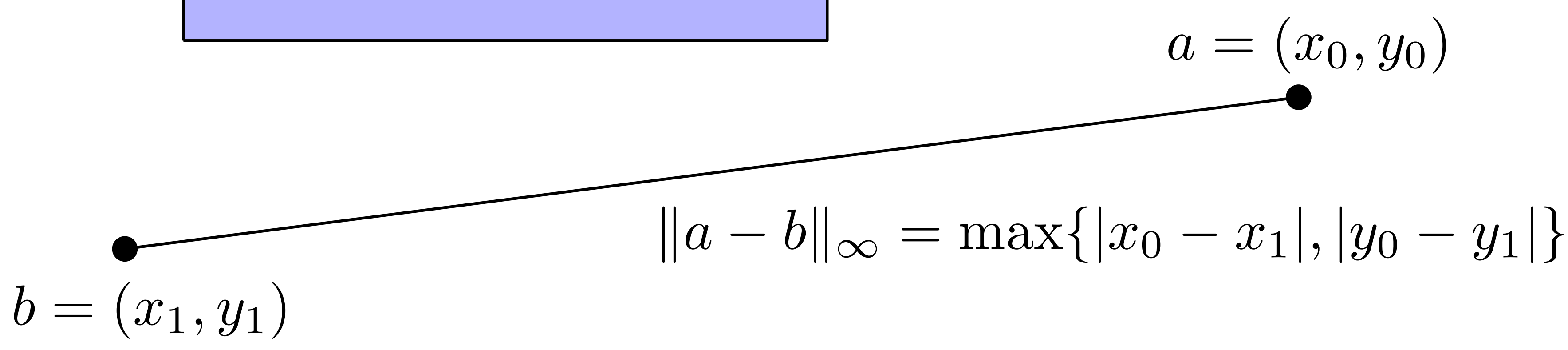

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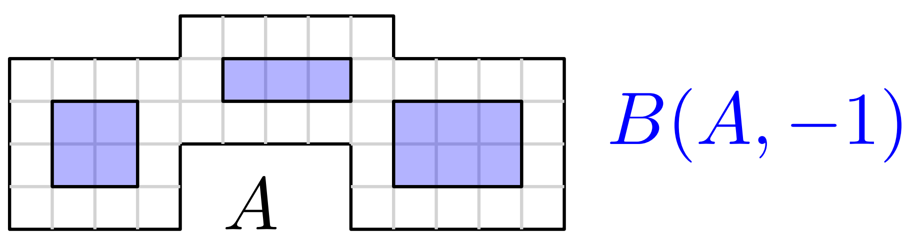

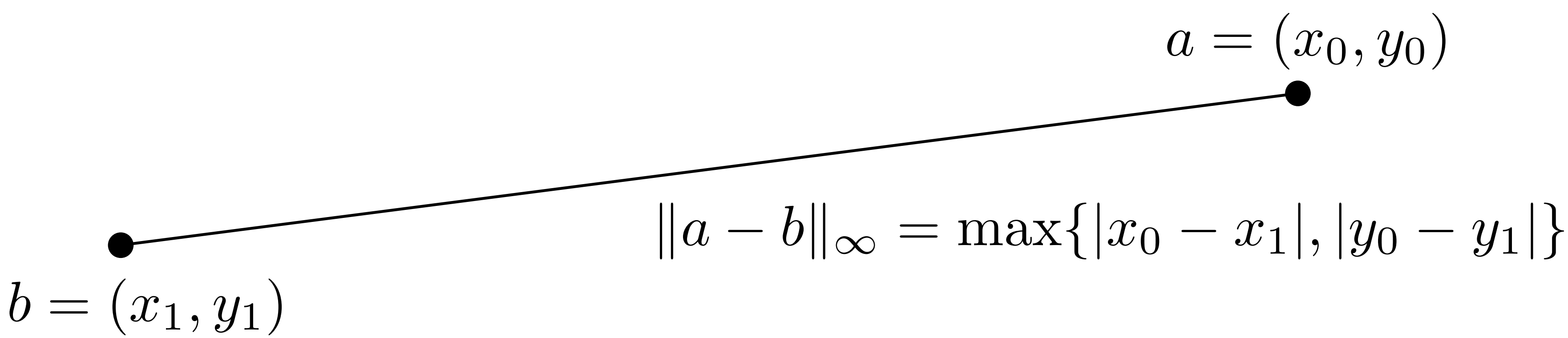

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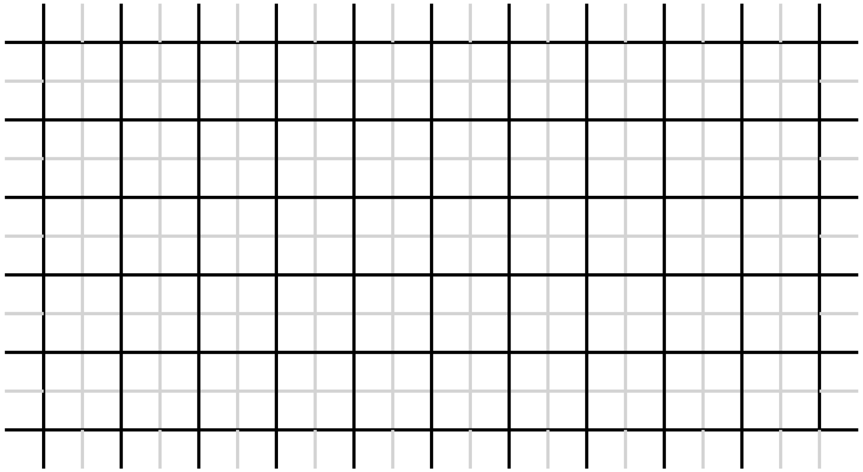

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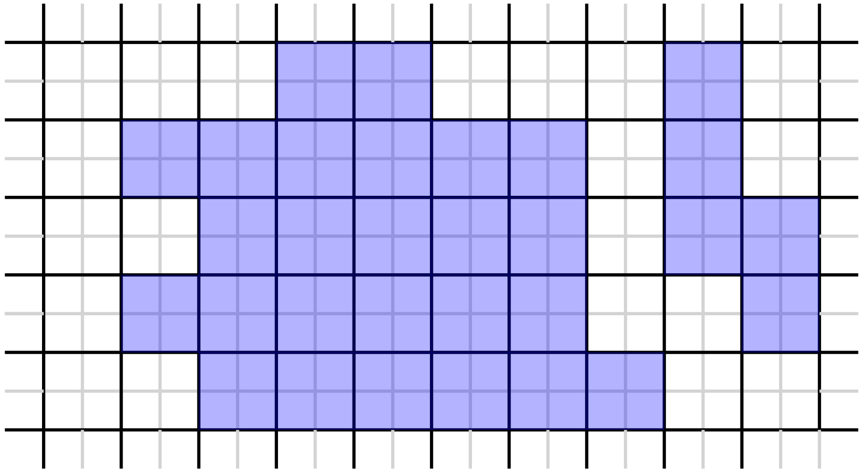

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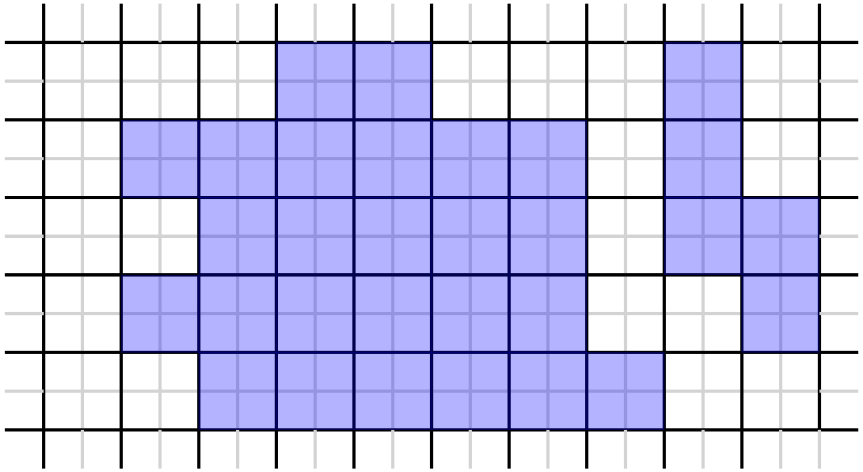

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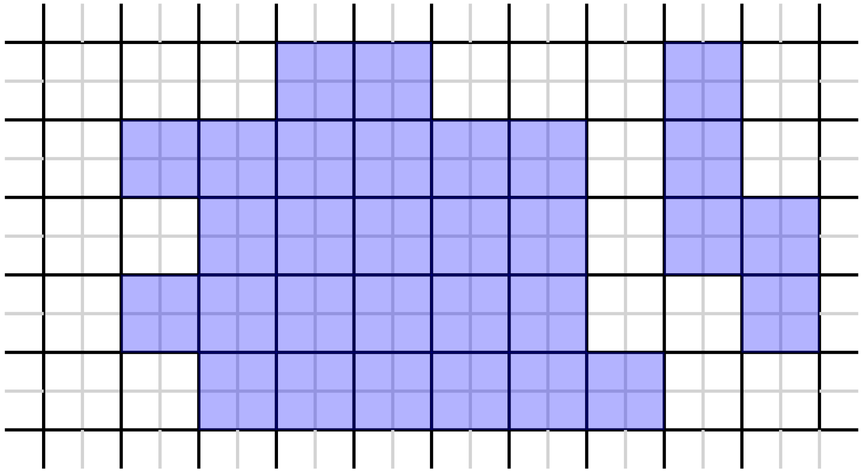

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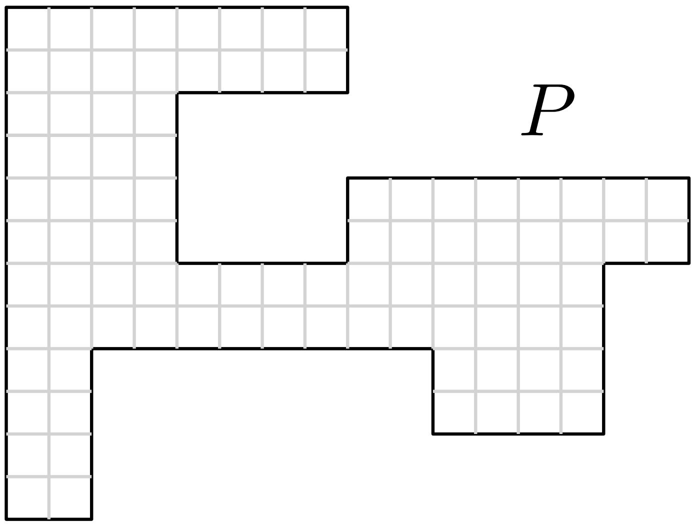

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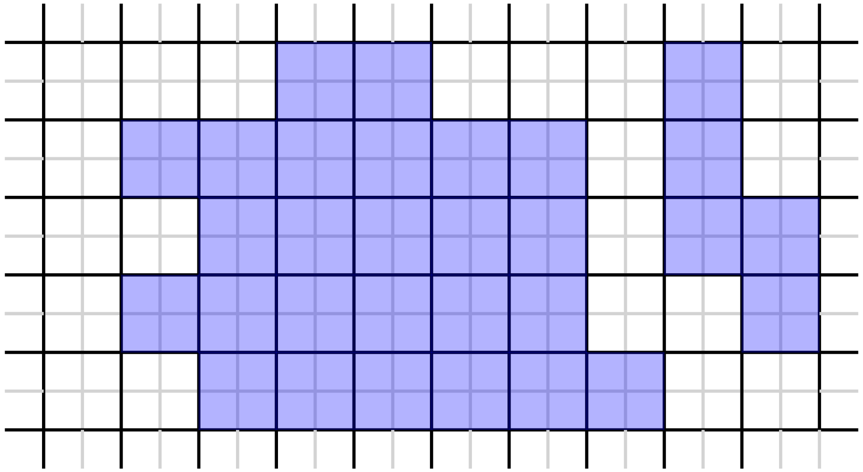

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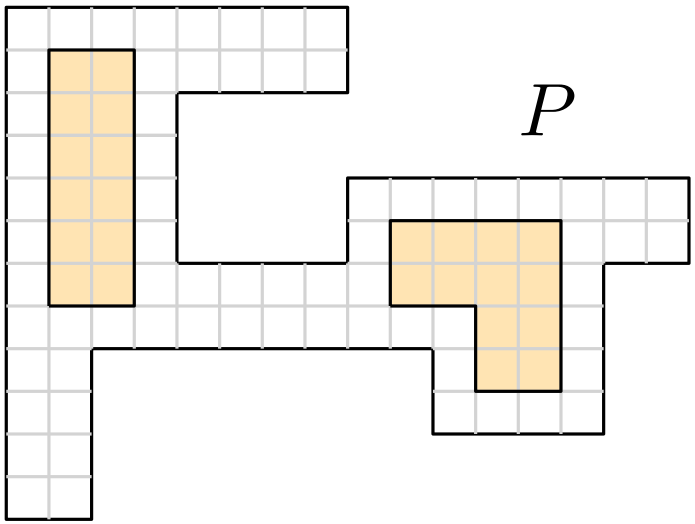

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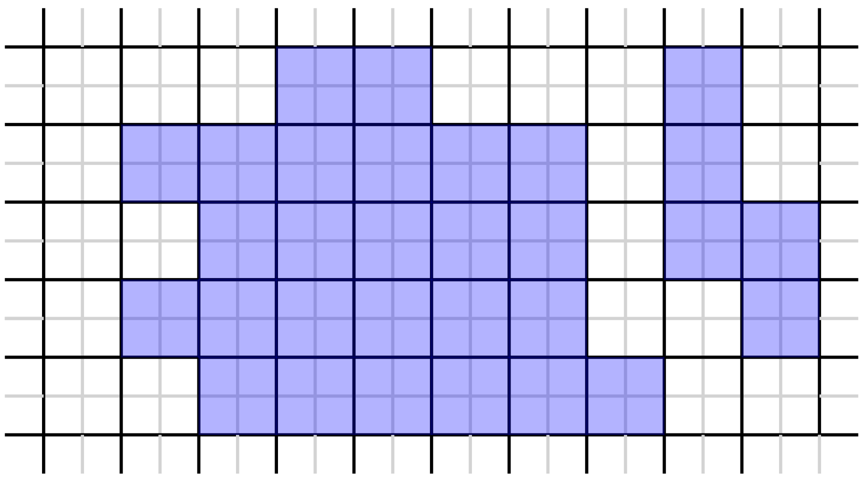

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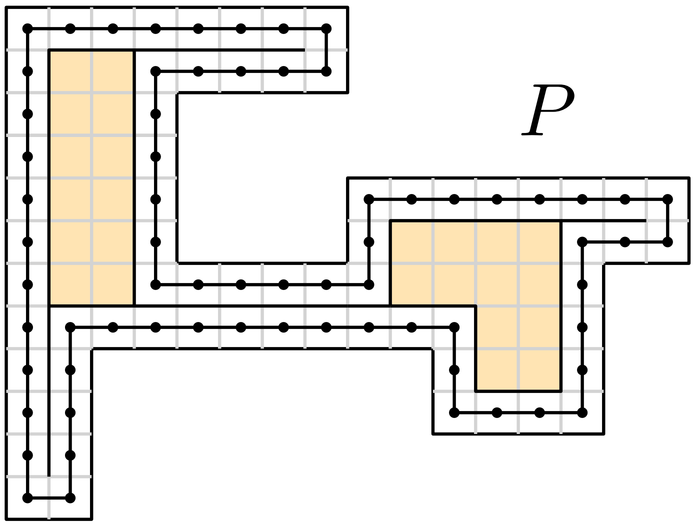

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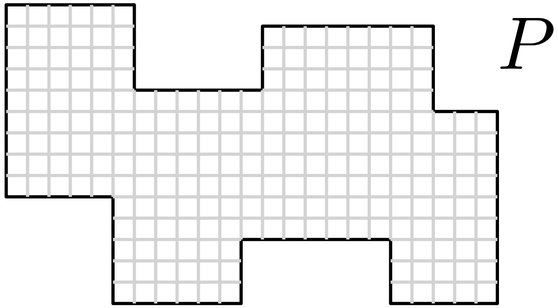

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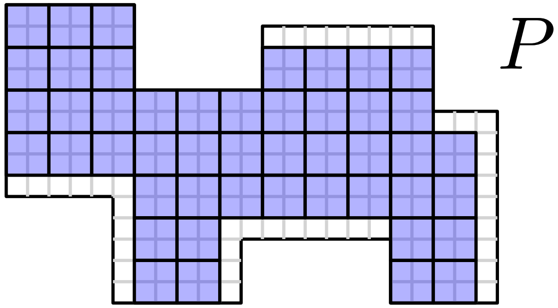

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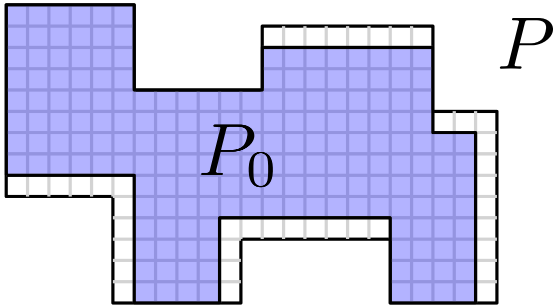

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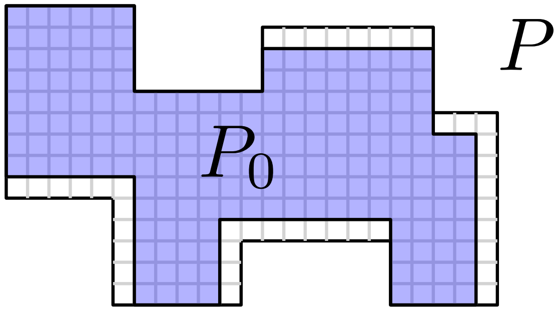

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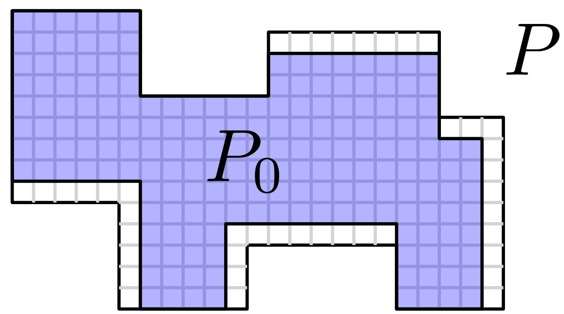

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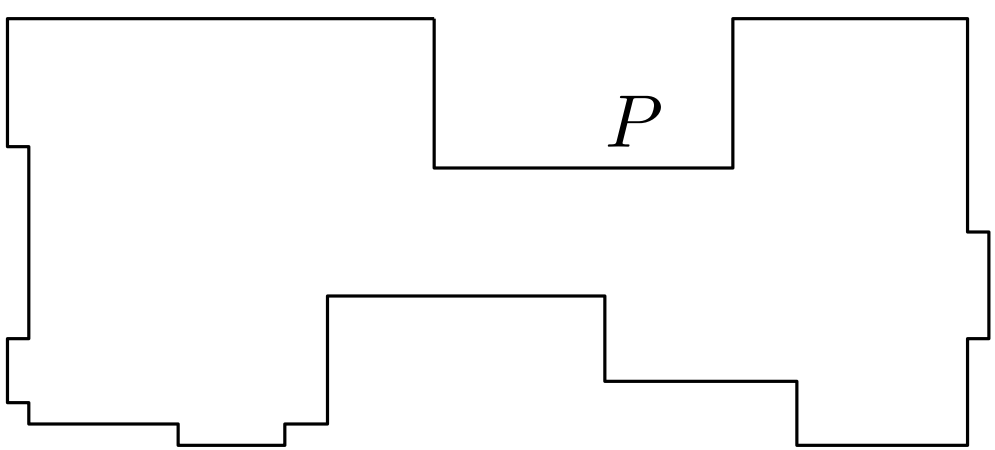

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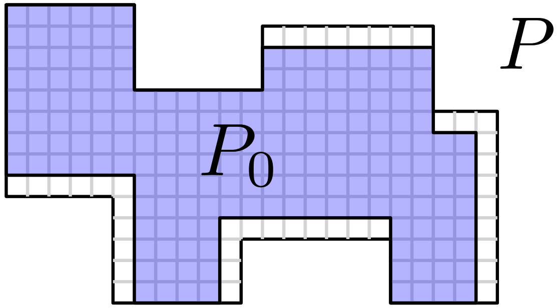

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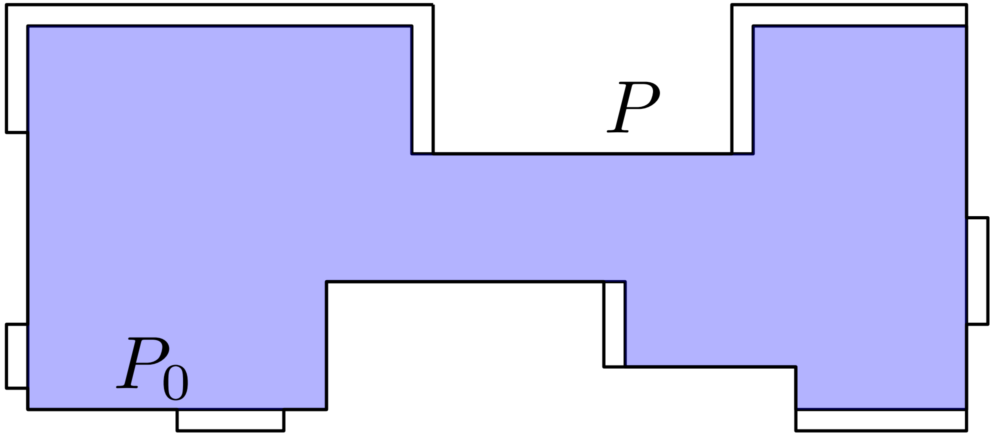

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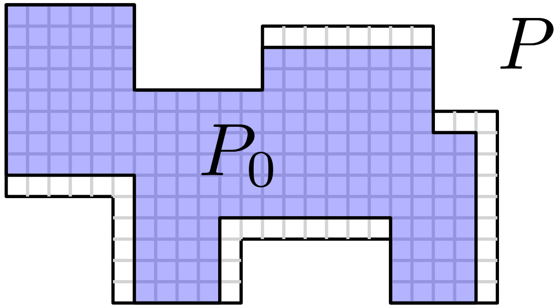

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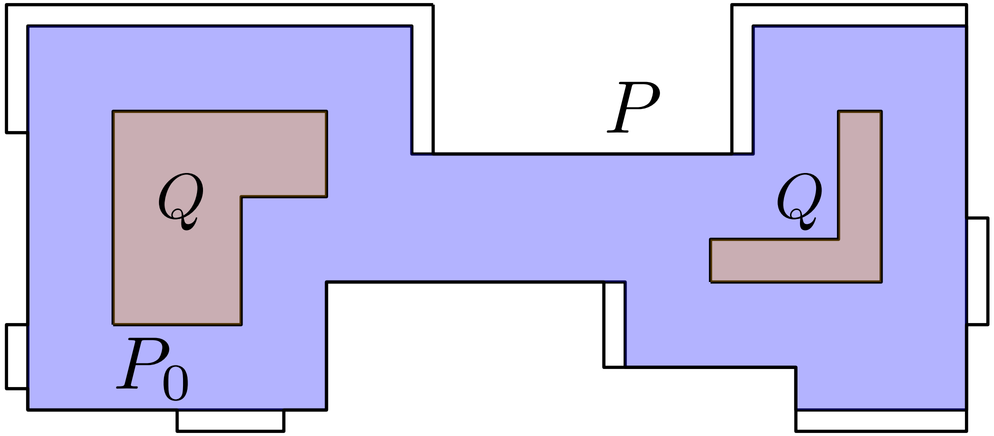

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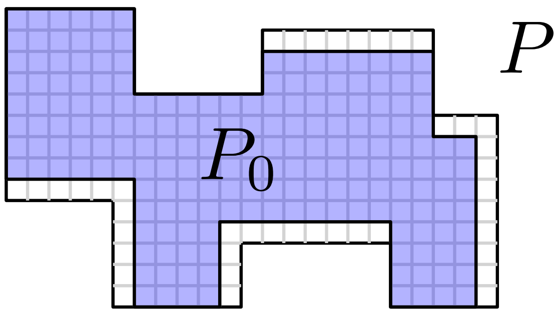

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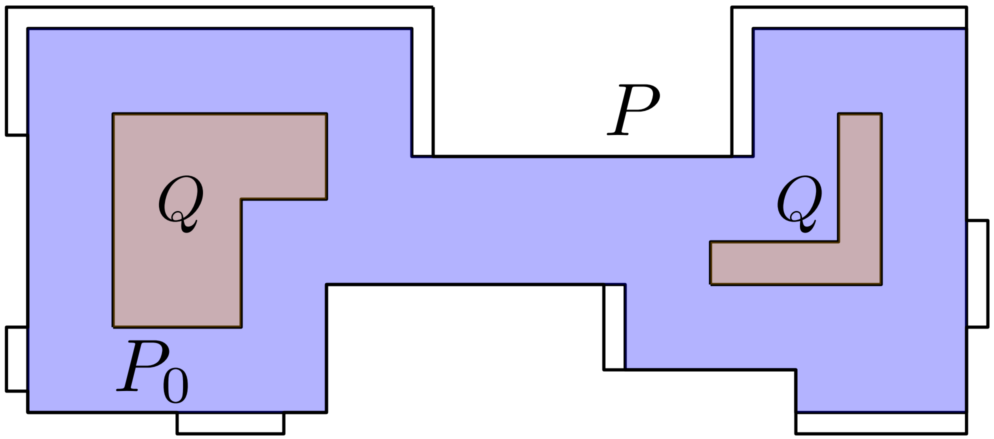

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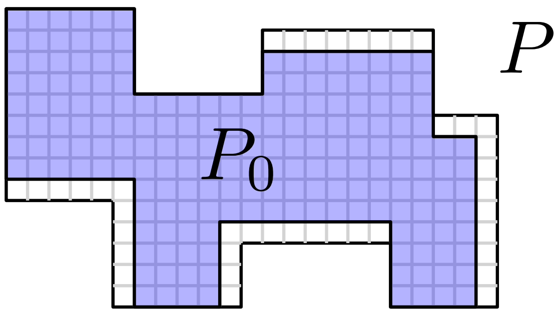

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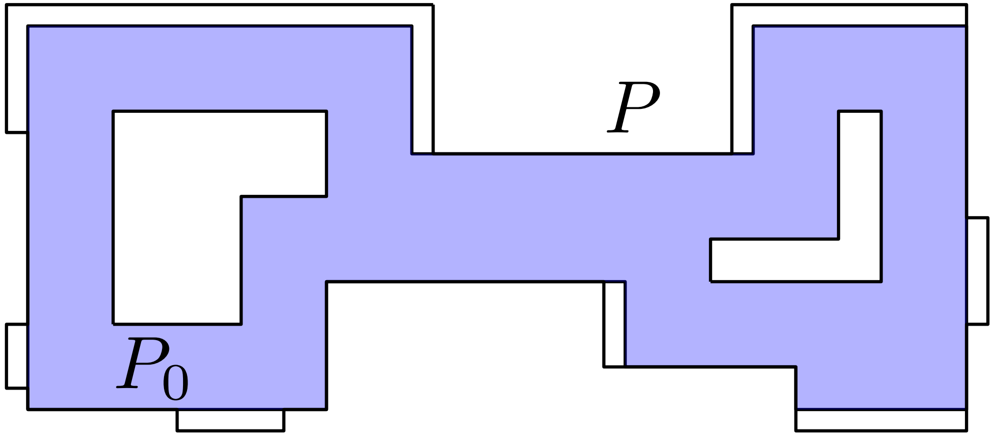

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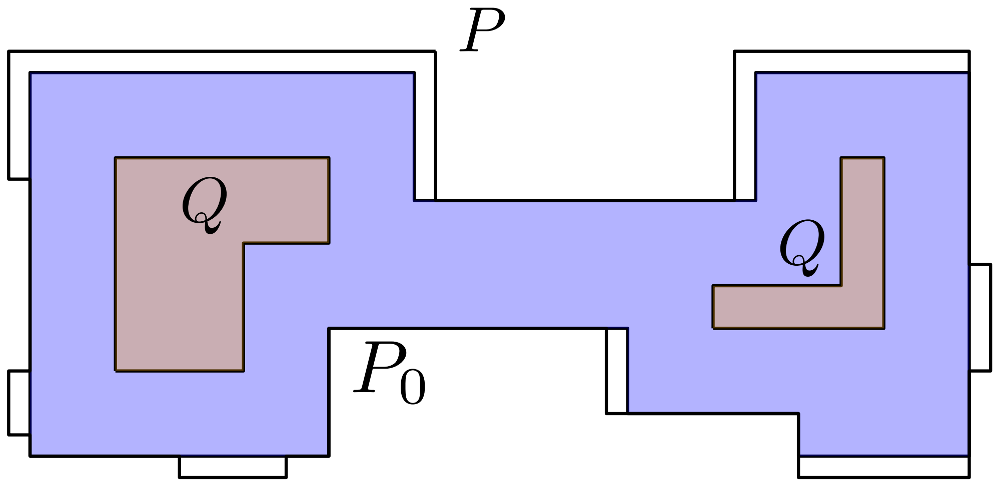

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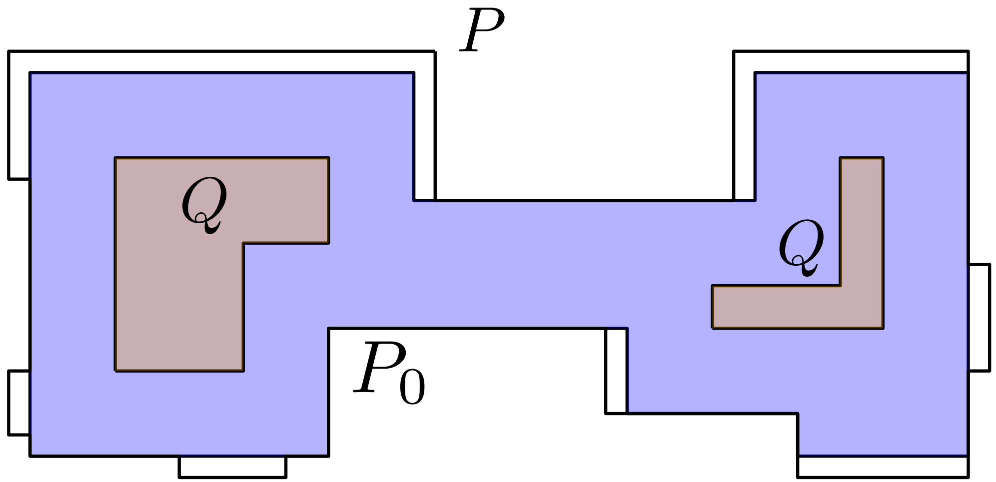

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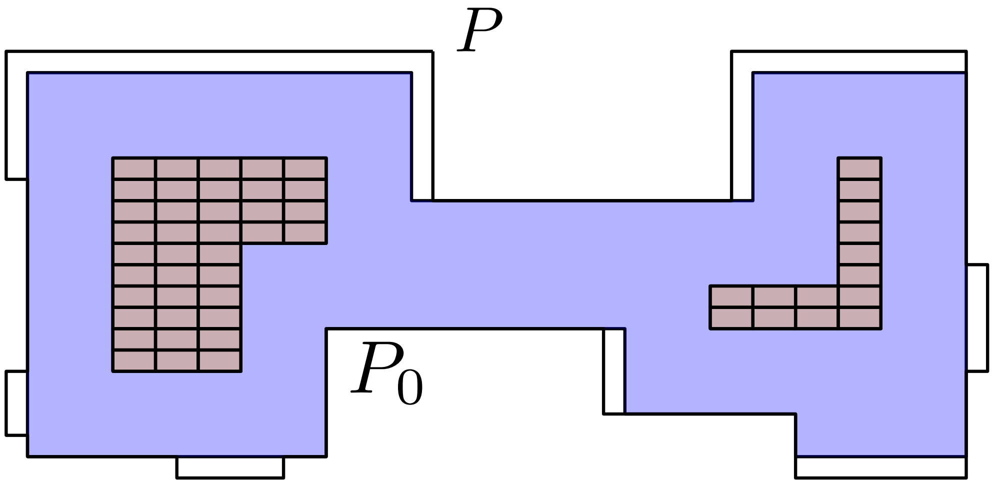

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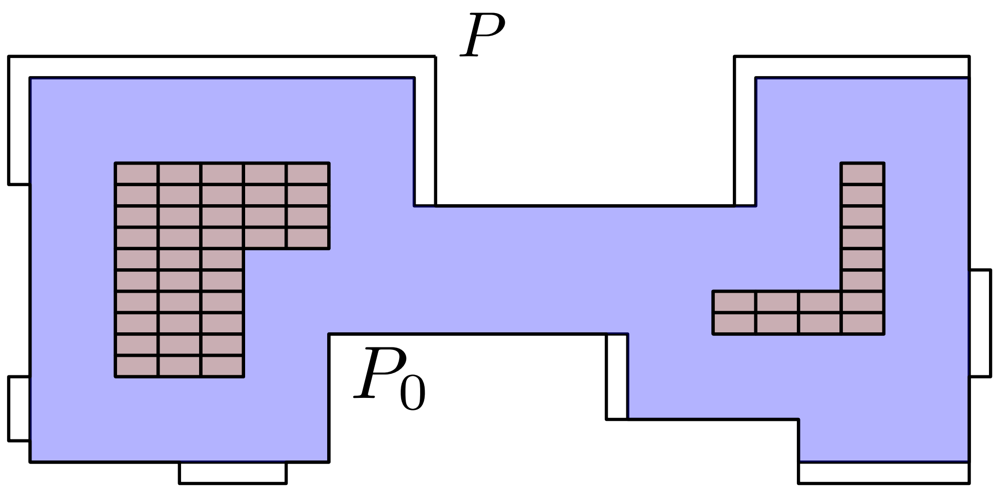

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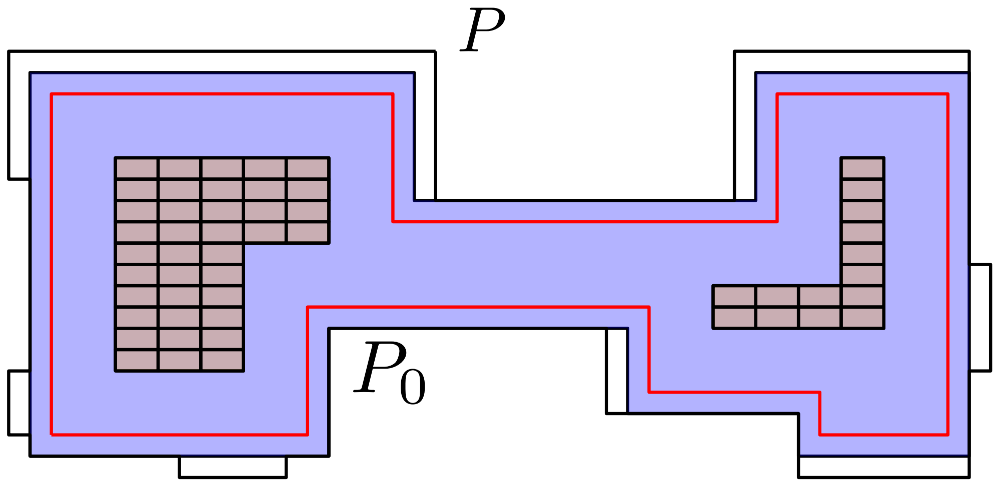

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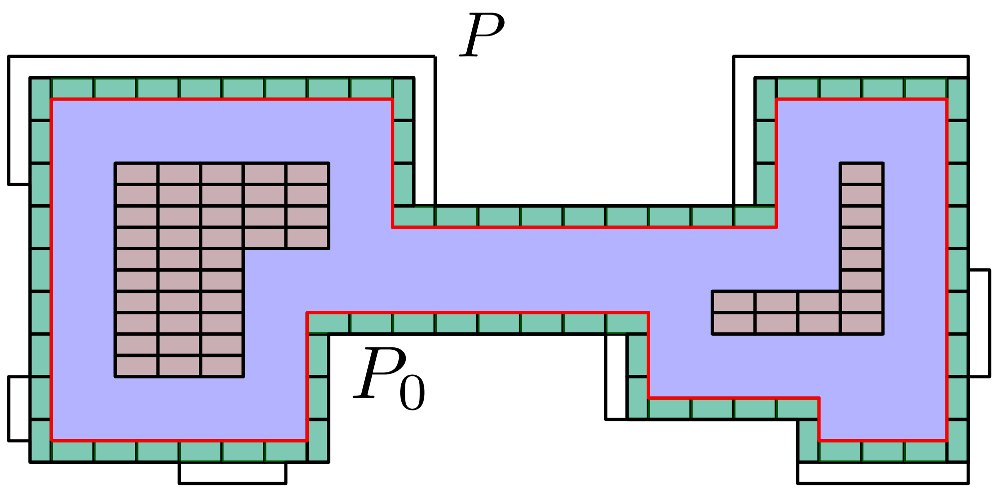

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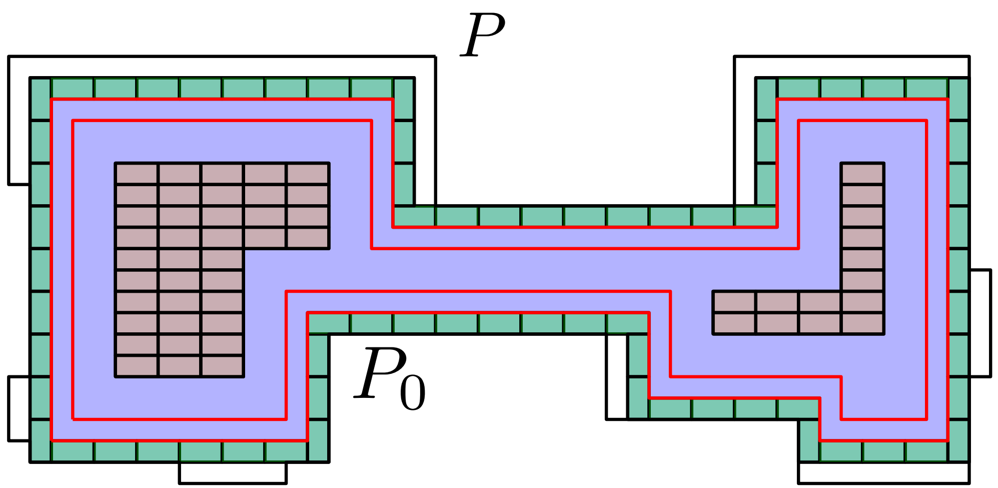

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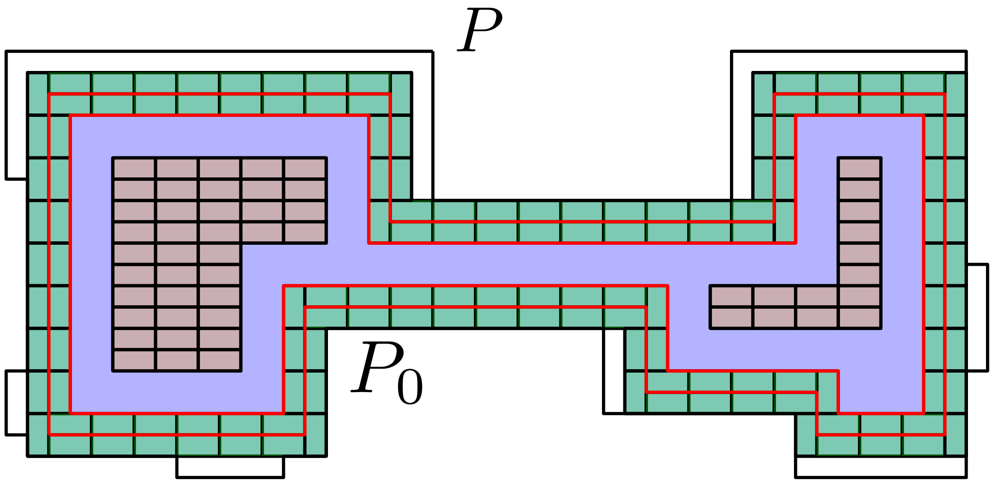

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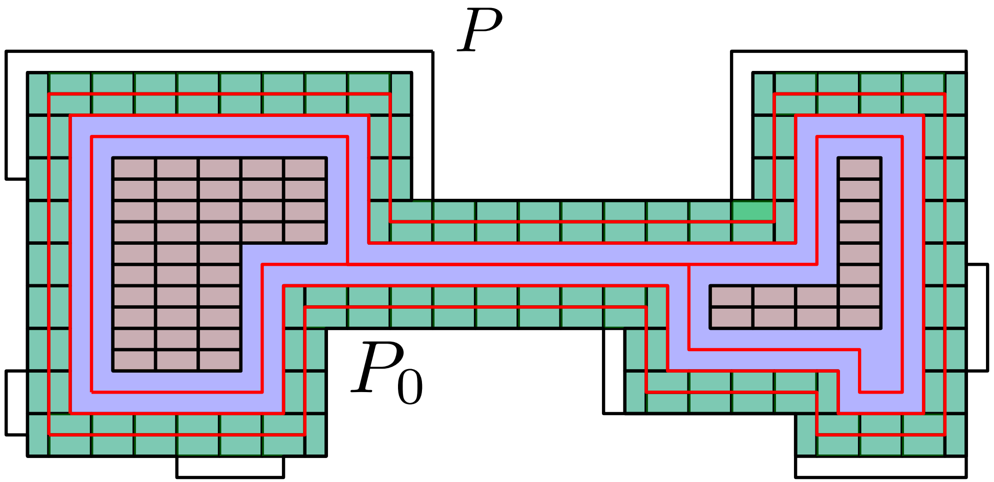

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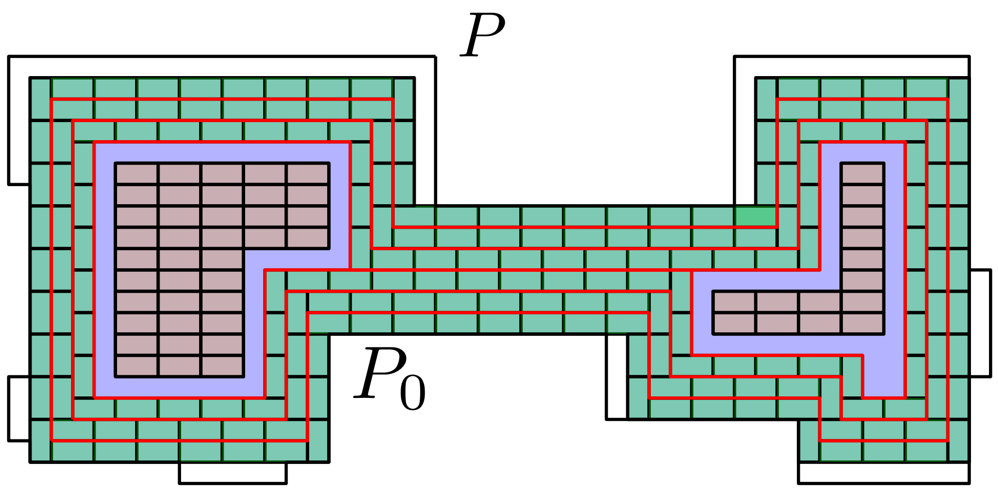

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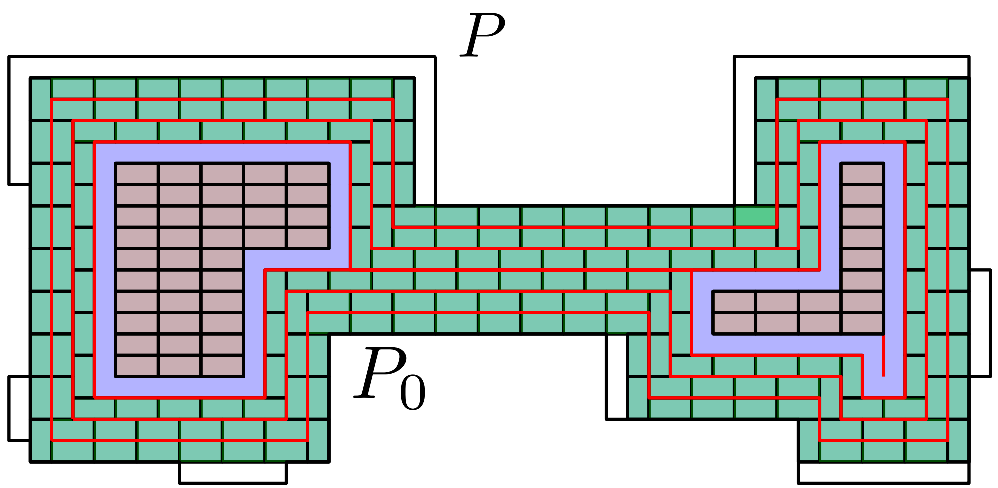

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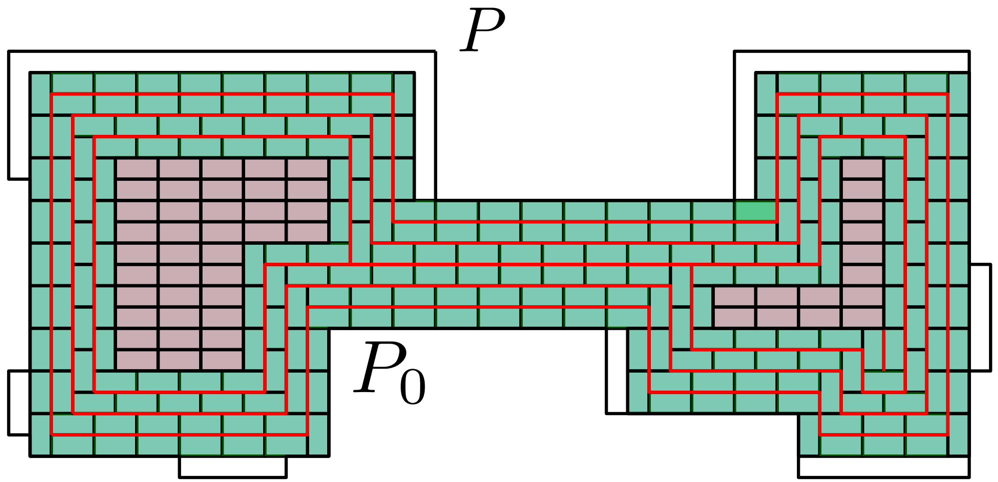

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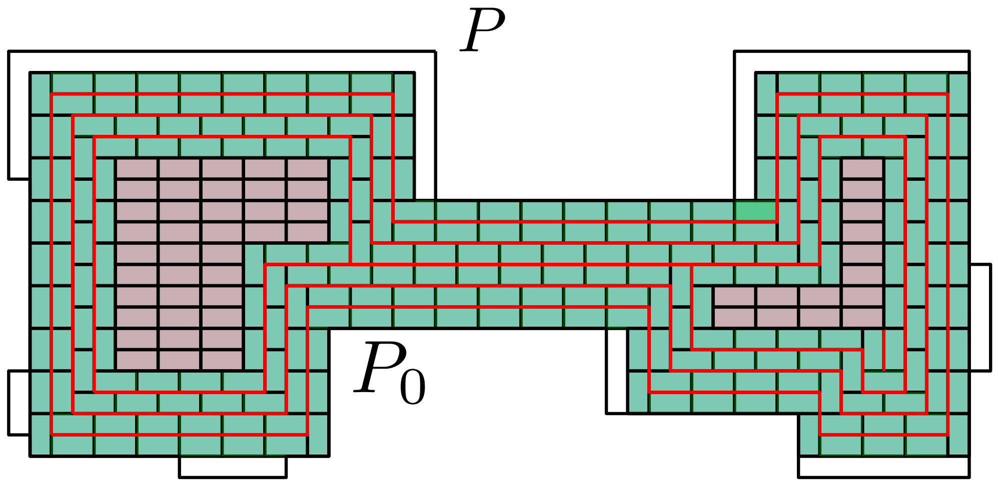

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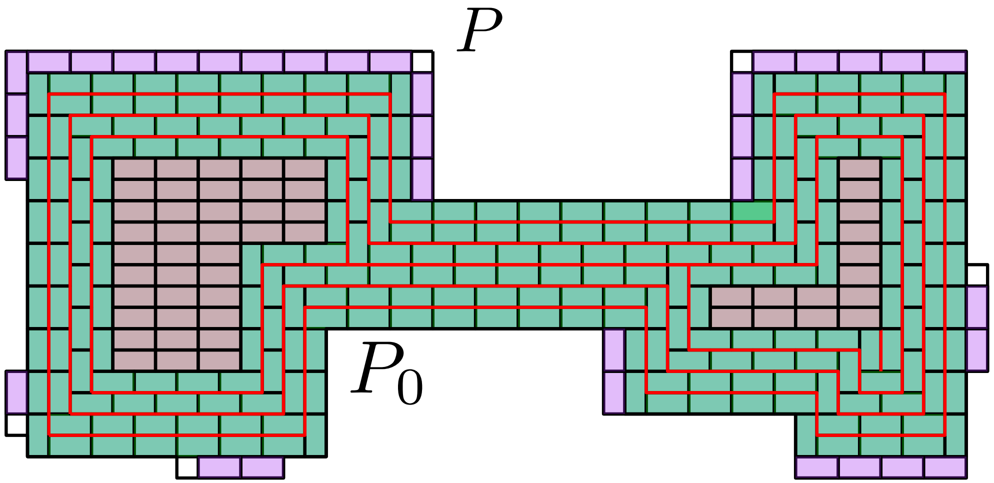

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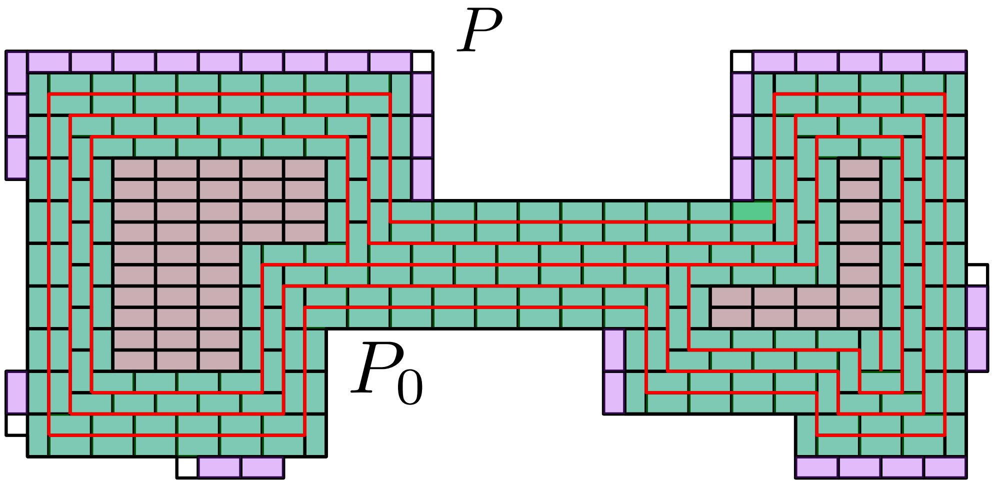

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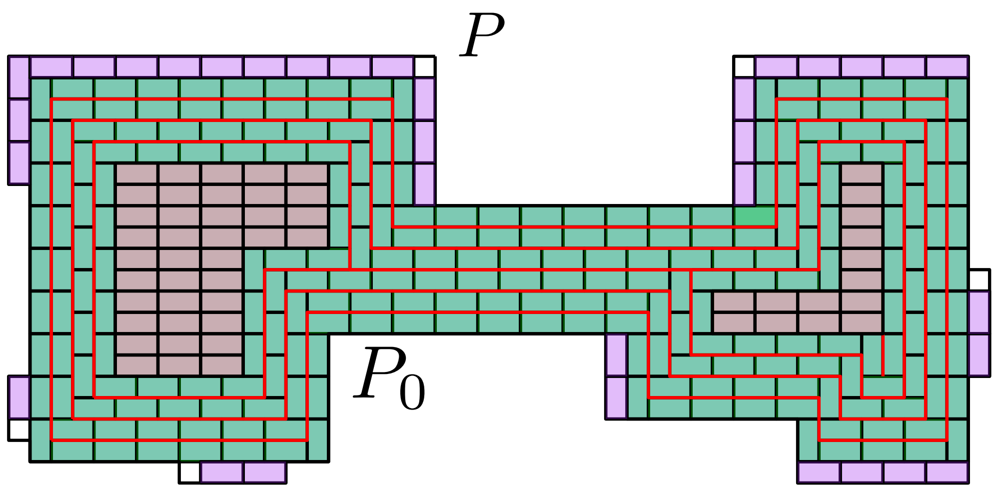

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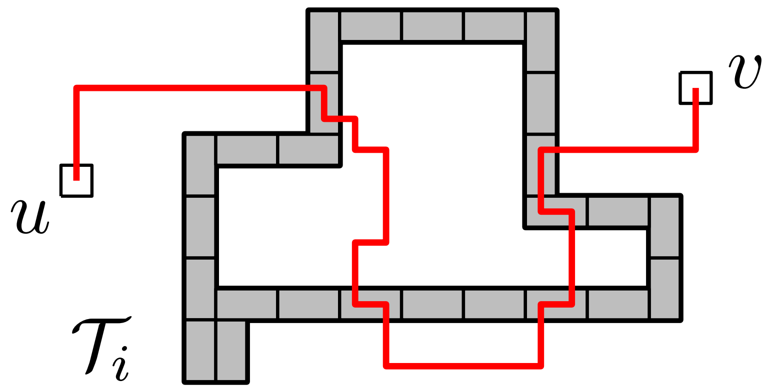

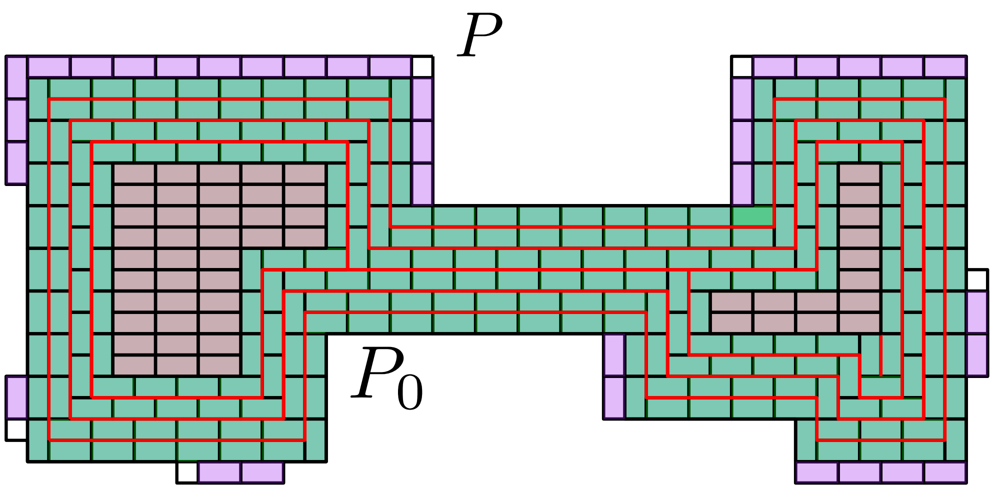

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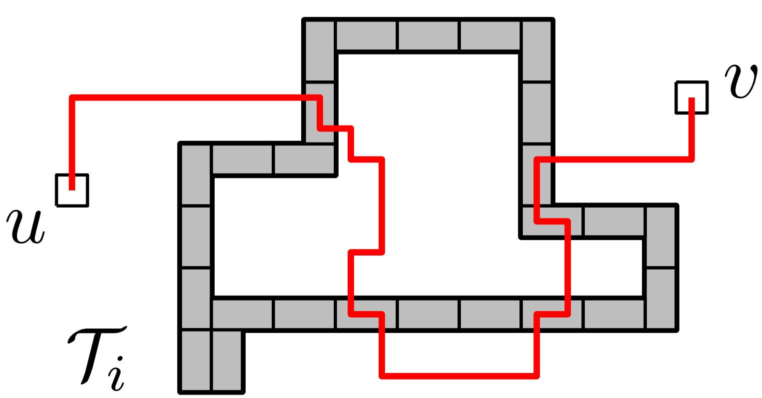

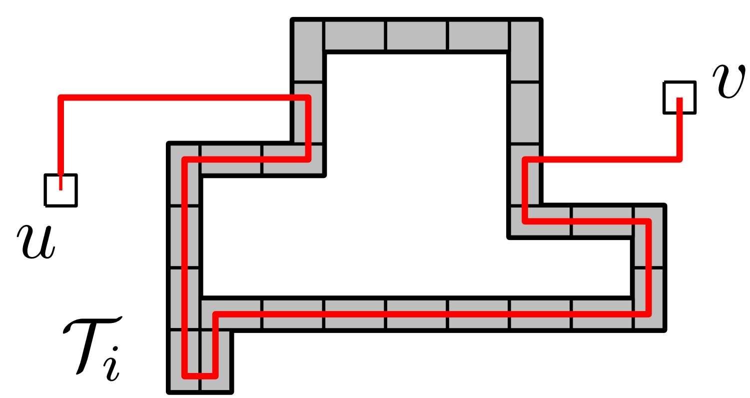

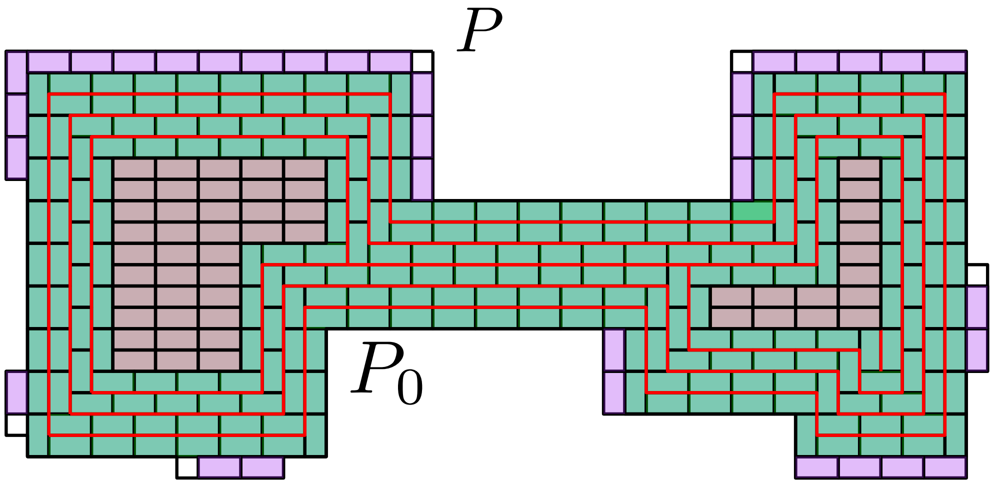

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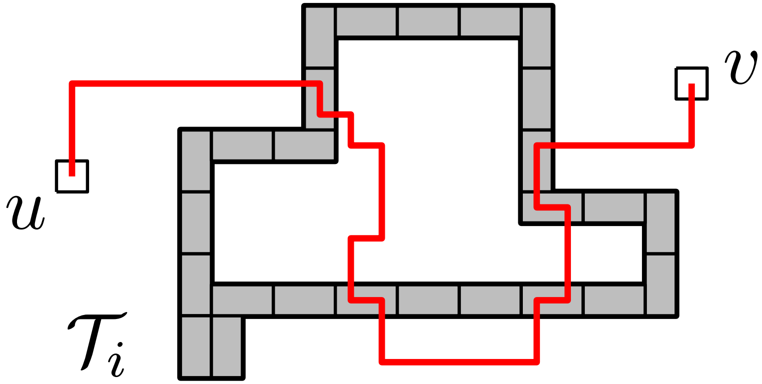

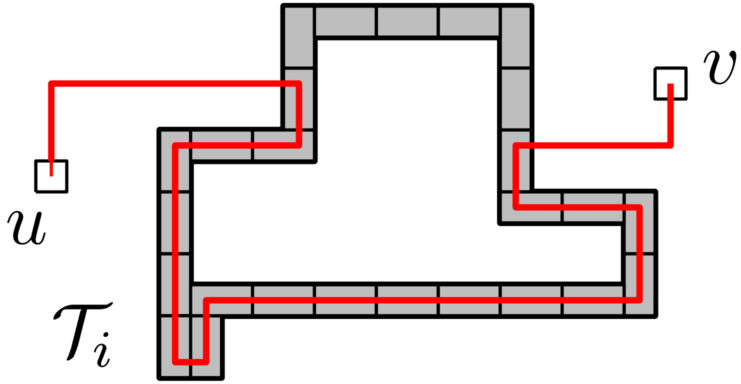

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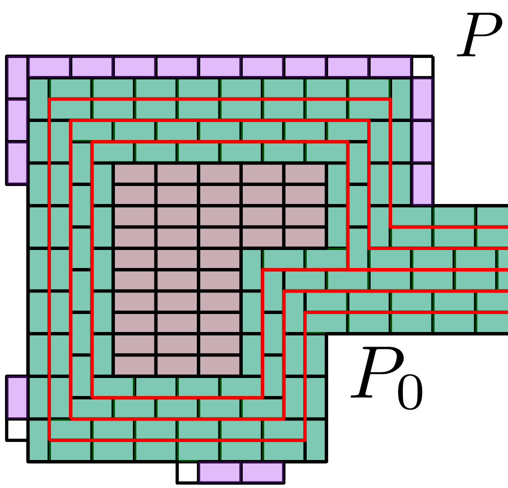

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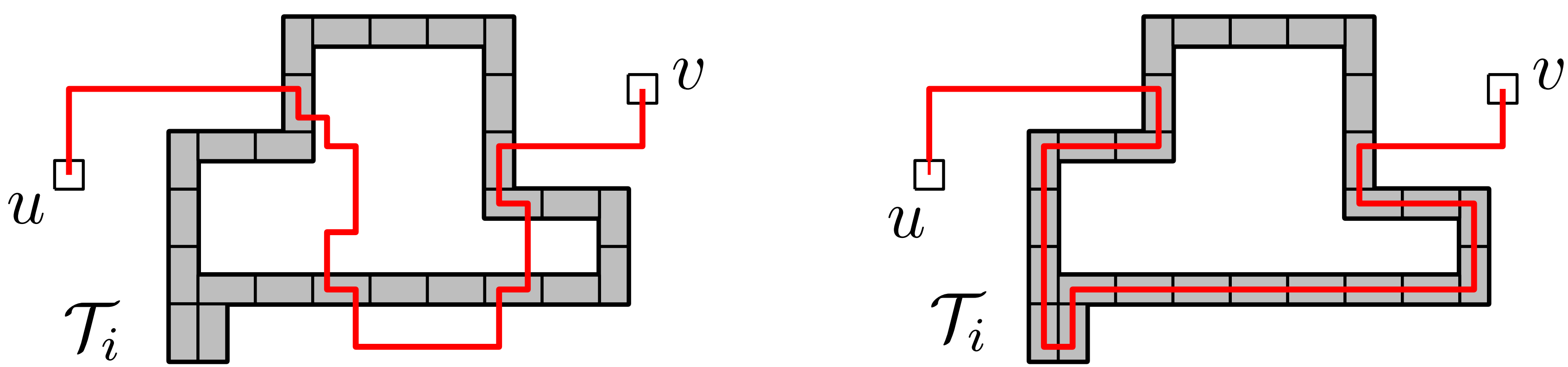
Importantly:
P has no holes $\Rightarrow$
u and v are both 'outside' each of the Hamiltonian cycles.



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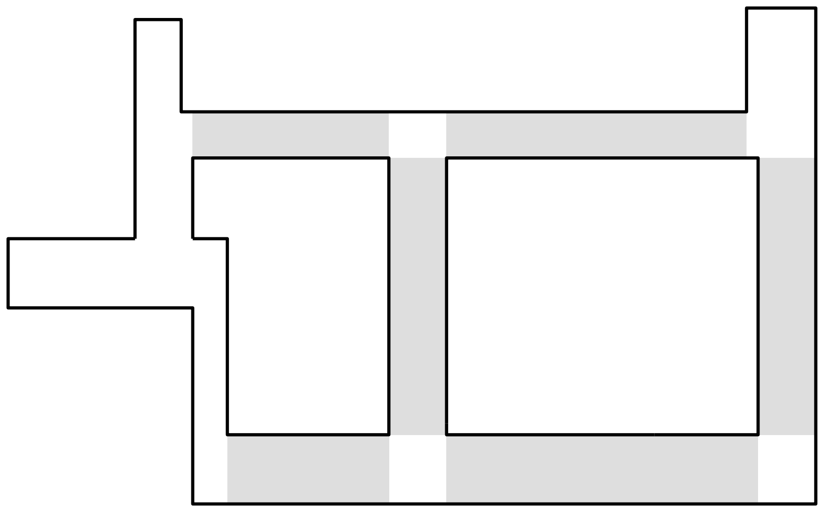

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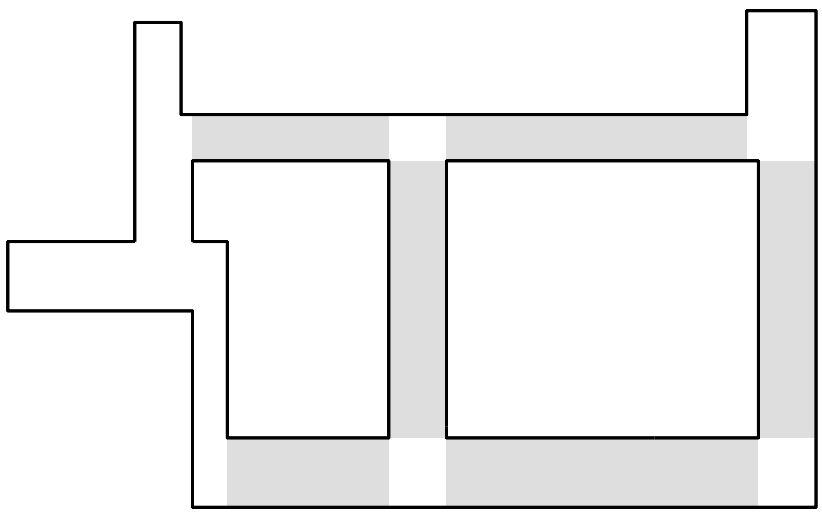

However, any point of $P' = P \setminus Q$ is of distance $O(n)$ to $\partial P'$

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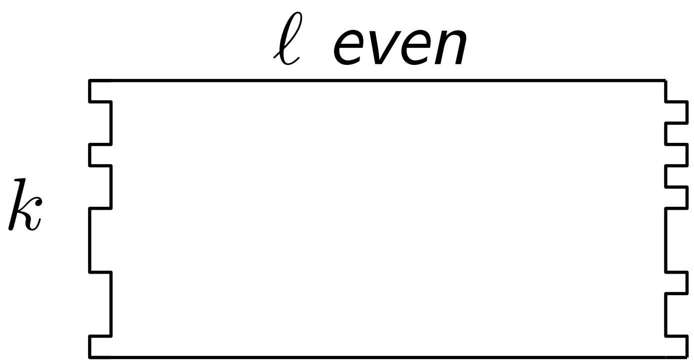

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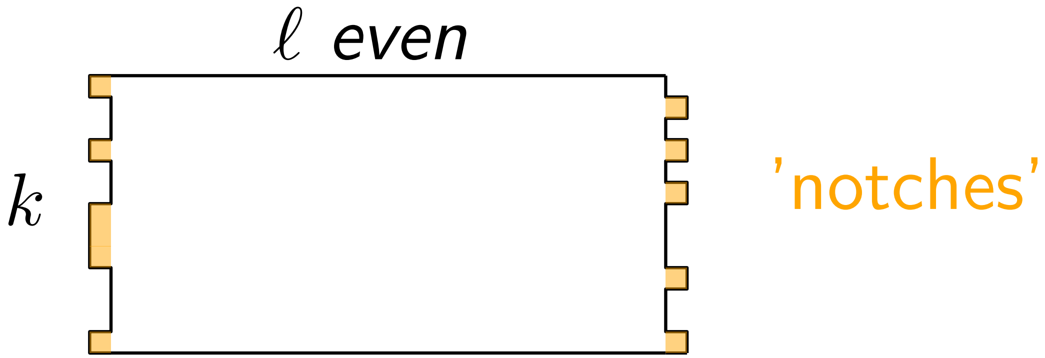

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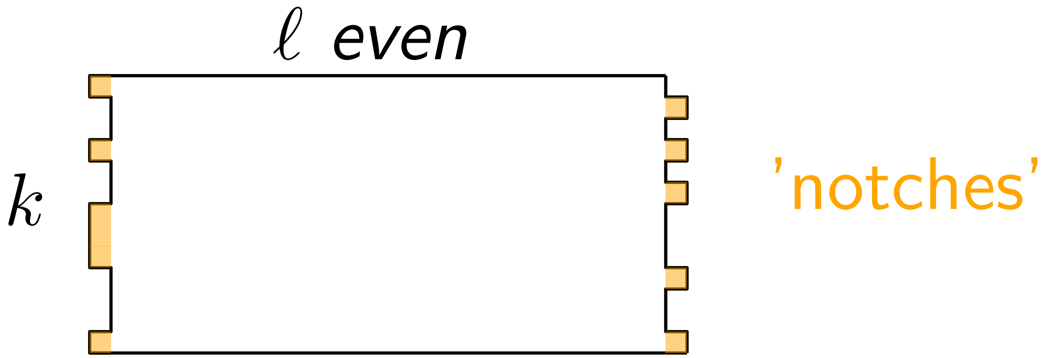

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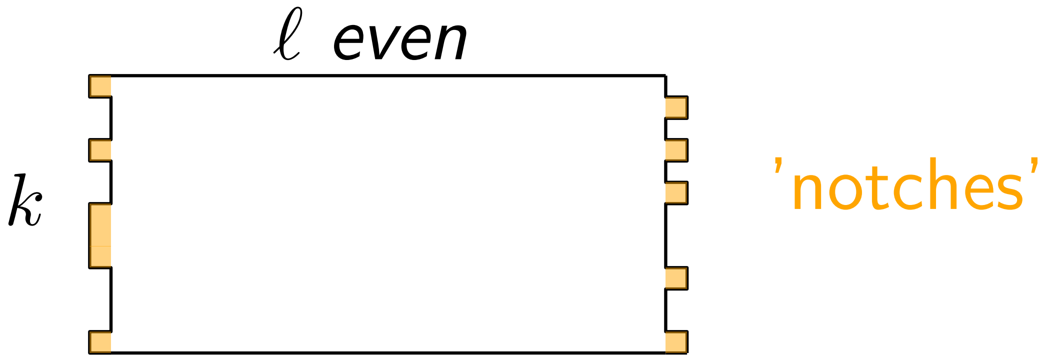

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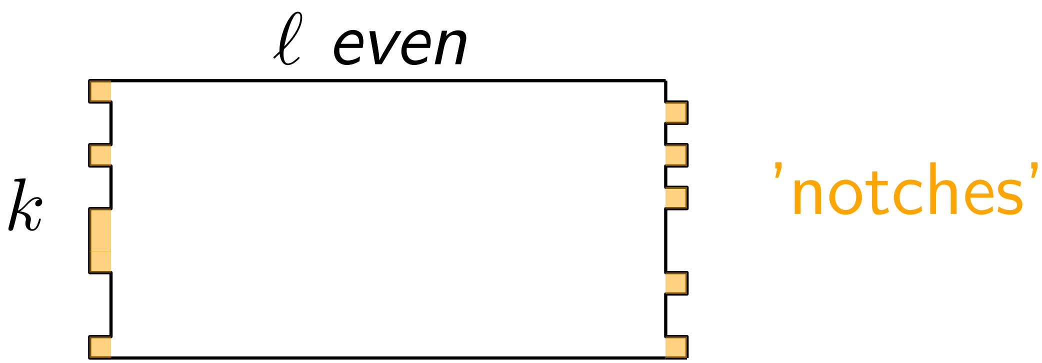

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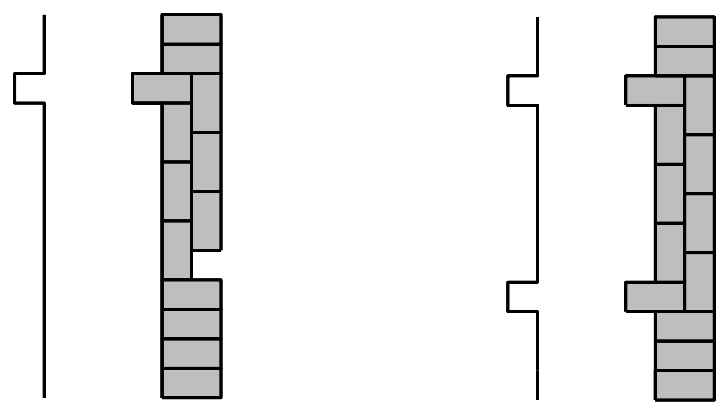

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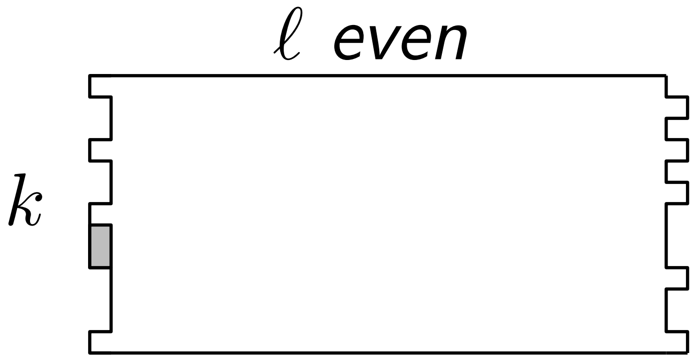

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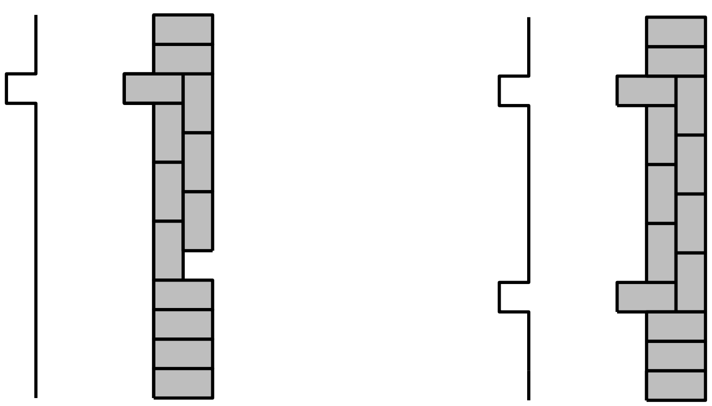

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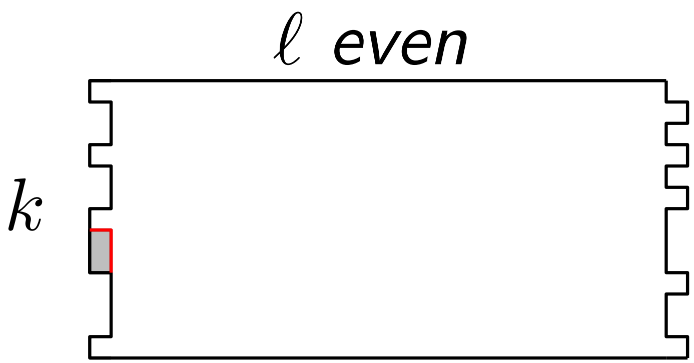

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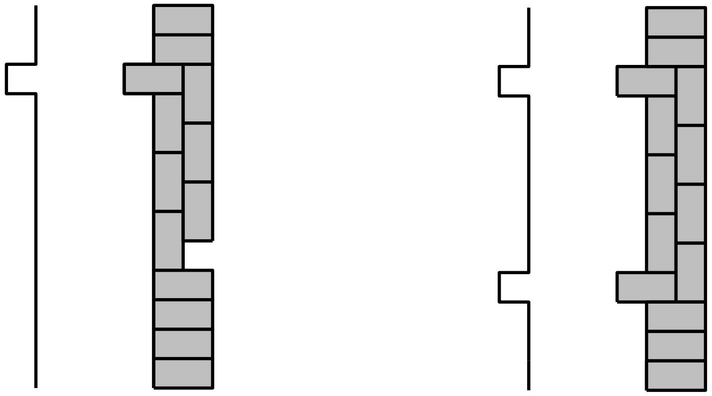

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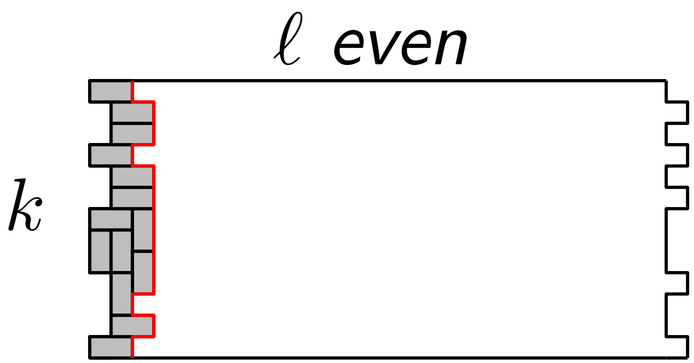

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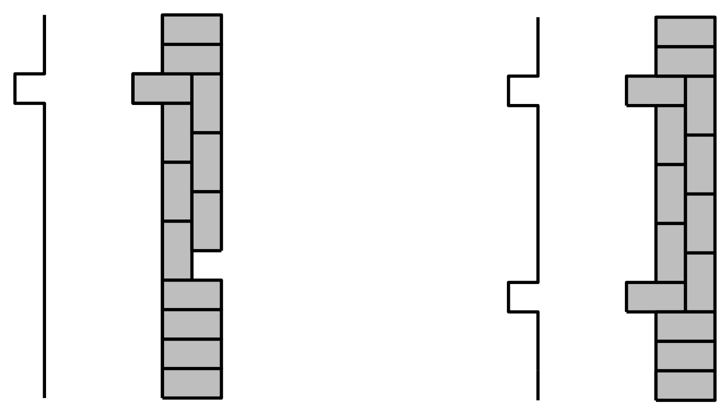

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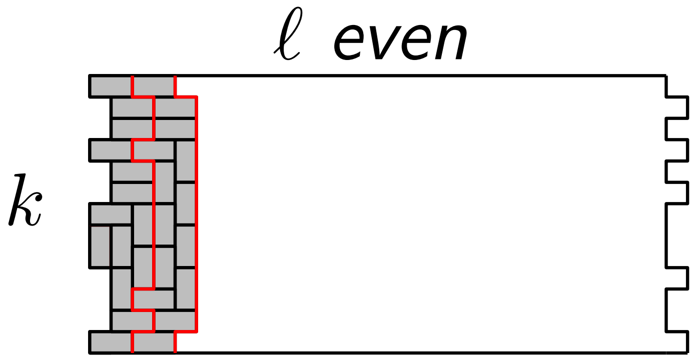

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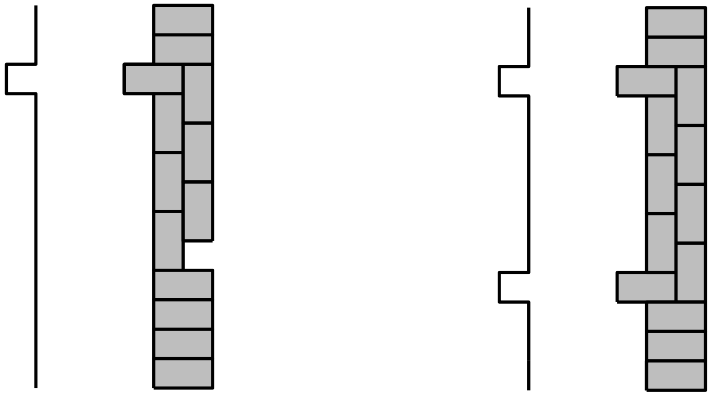

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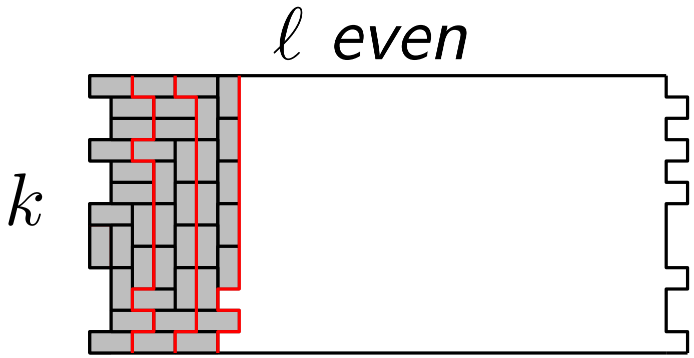

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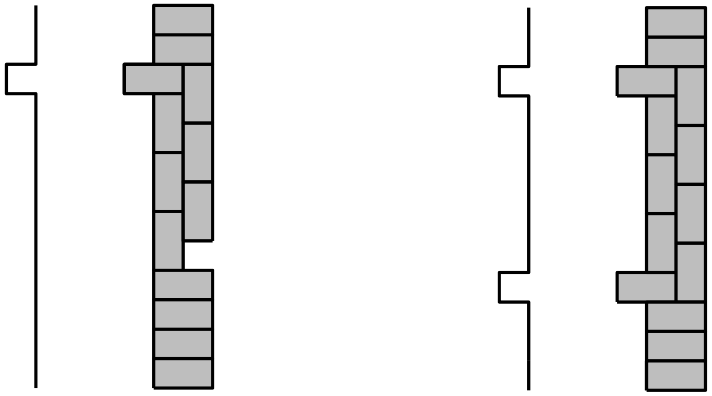

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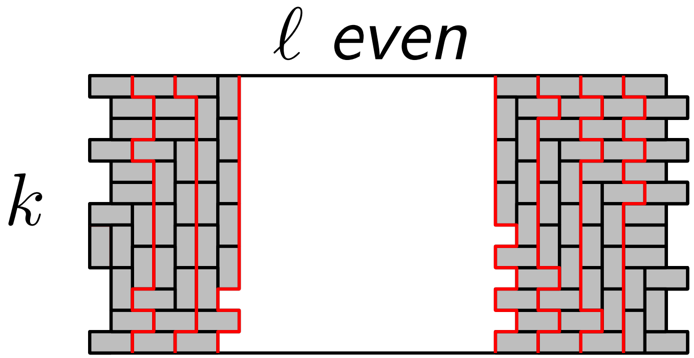

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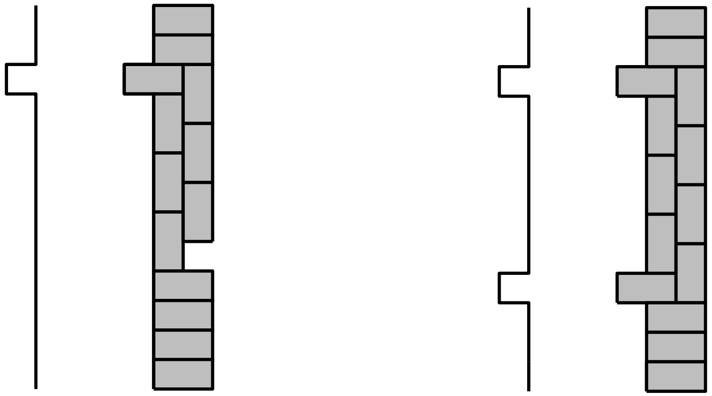

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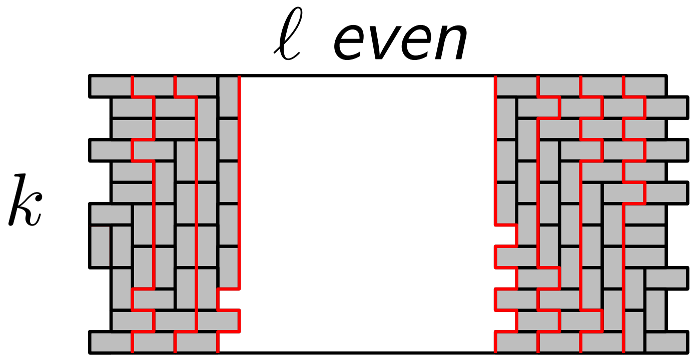

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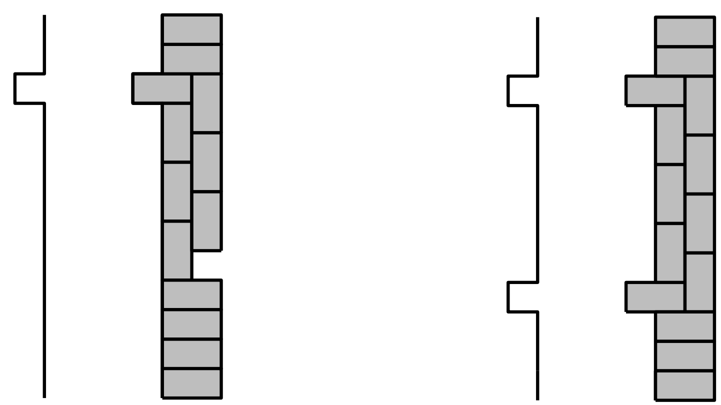

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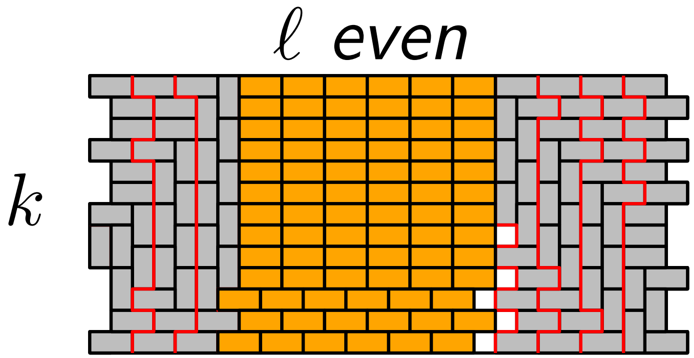

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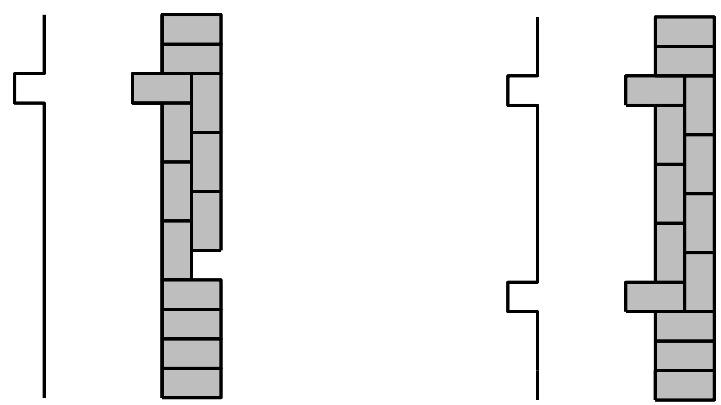
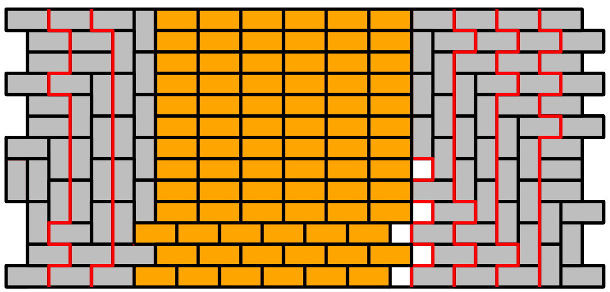


Structural Result 2: Shortening Pipes

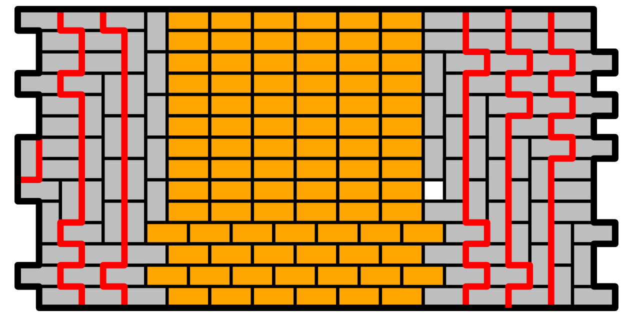
Consider 'pipes' of the form:

ℓ *even*

k

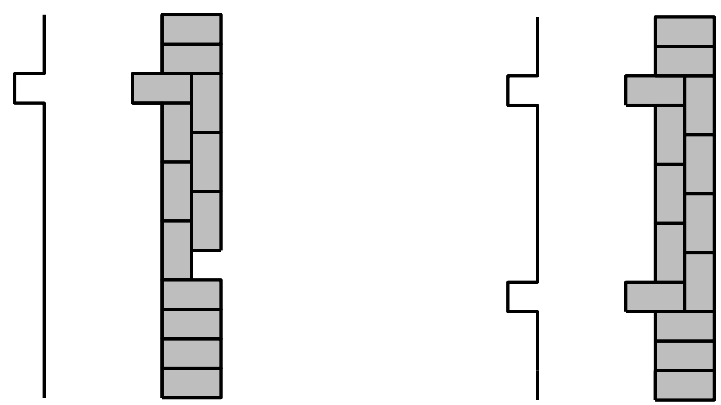


Now fill in horizontal dominos

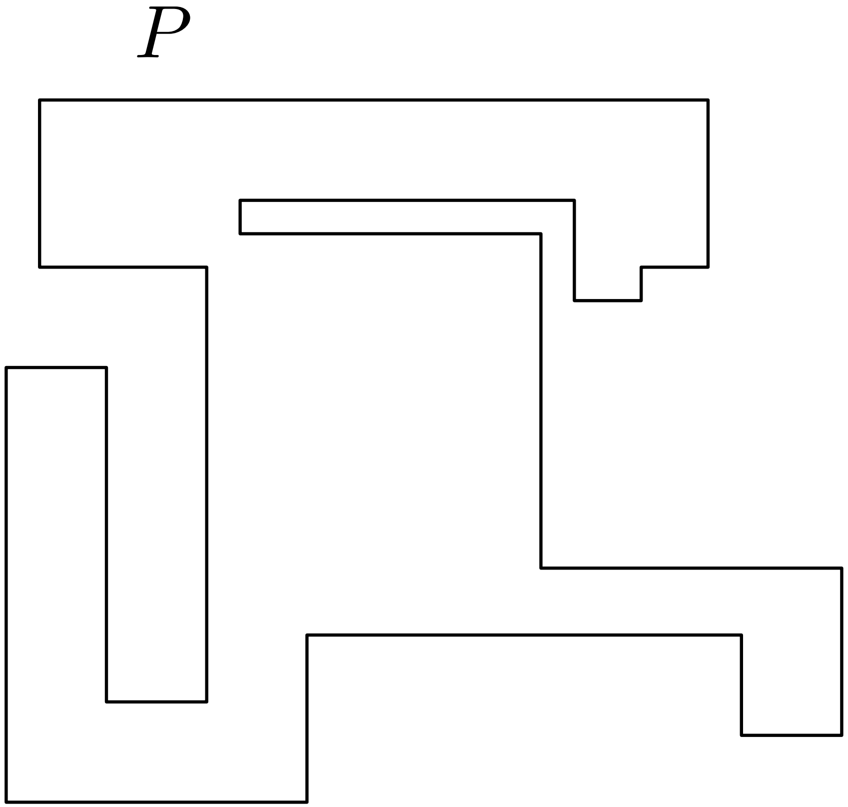


Color black and white in chessboard fashion with b black cells and w white cells. Assume $b \geq w$.

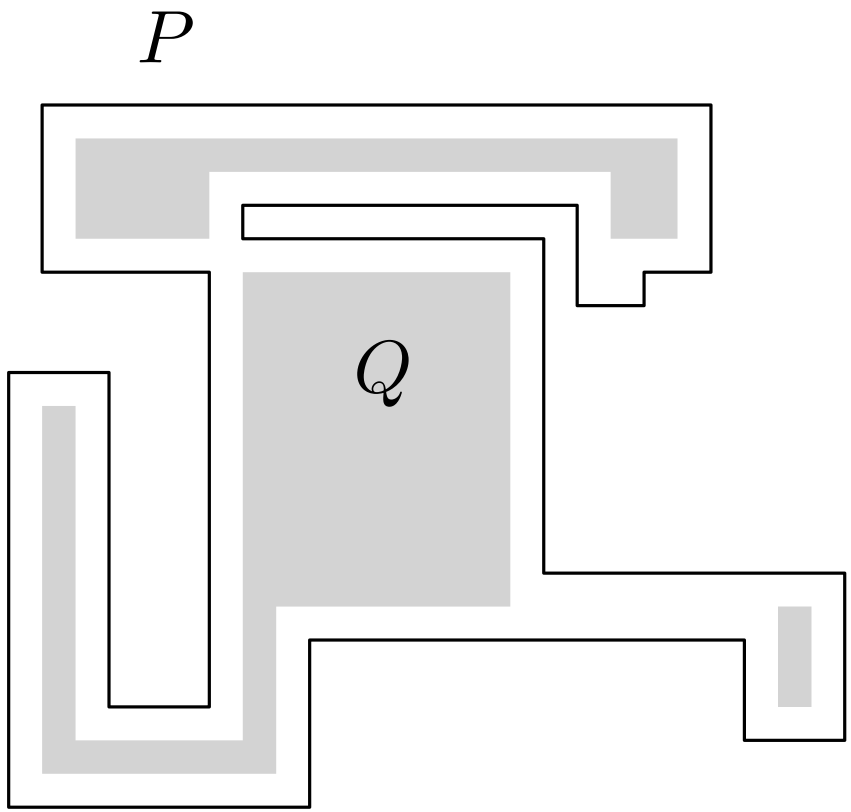
Lemma. *If $\ell \geq 2k$, then the number of uncovered cells in a maximum domino packing is $b - w$*



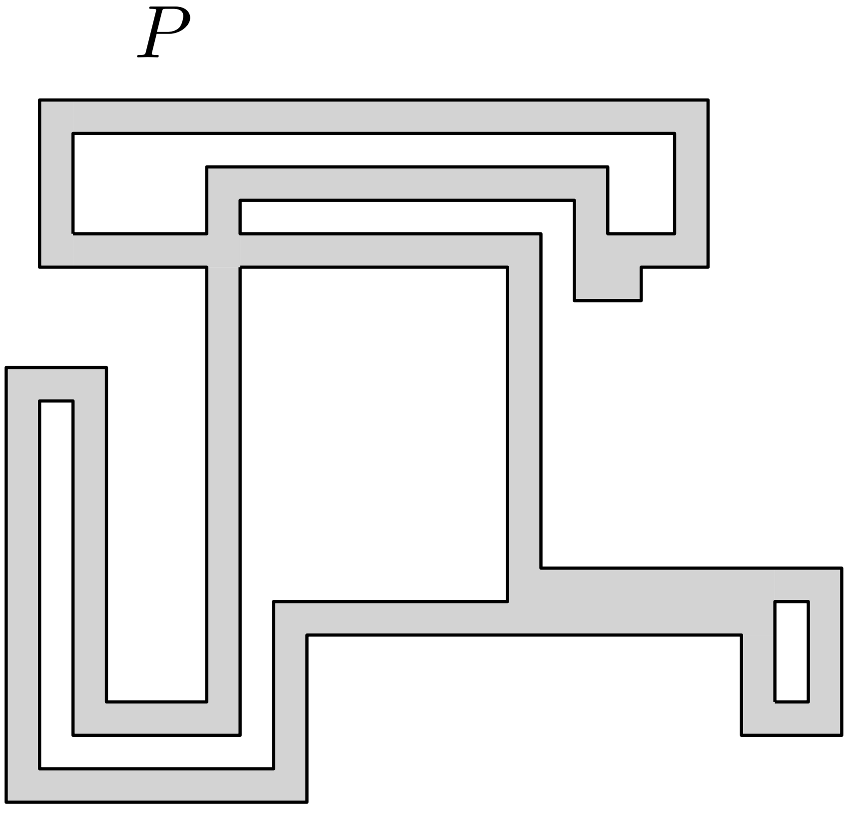
The final reduction



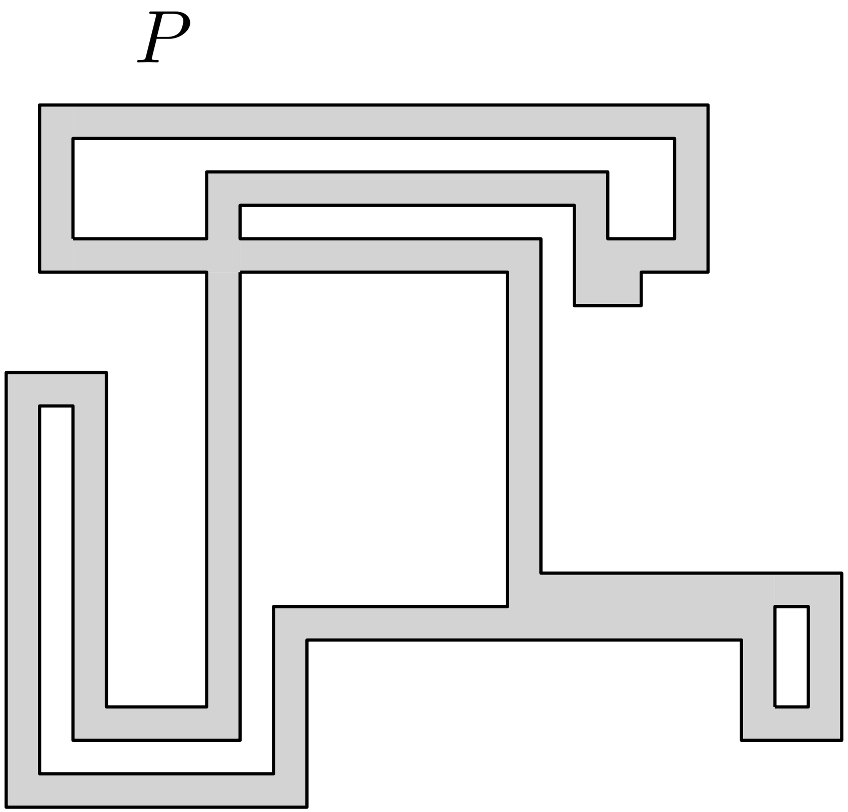
The final reduction



The final reduction

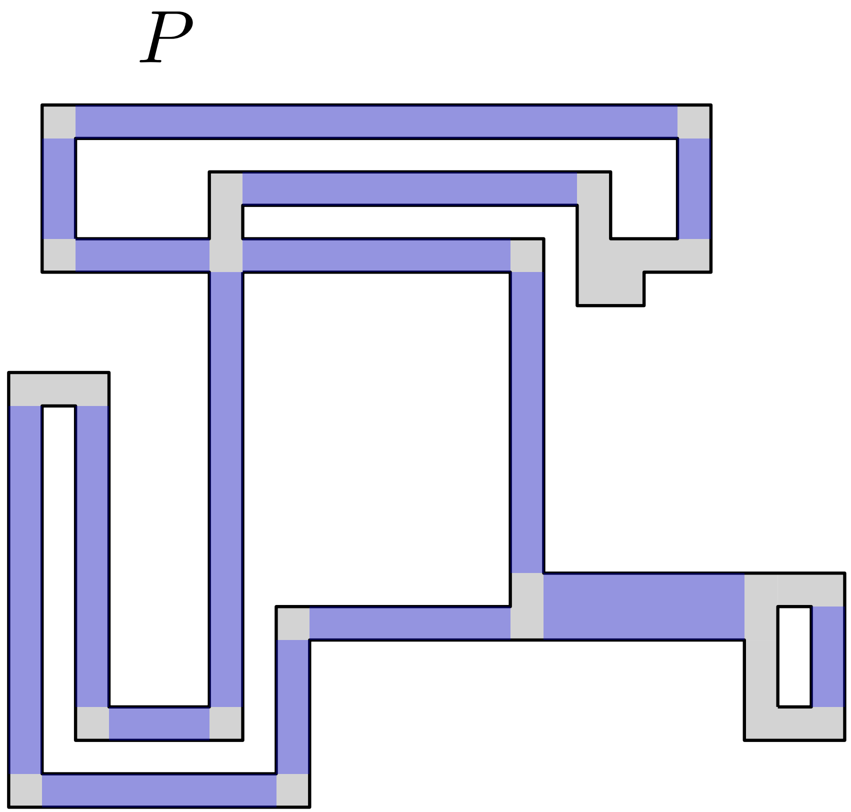


The final reduction



Find all pipes of length at least twice their width.

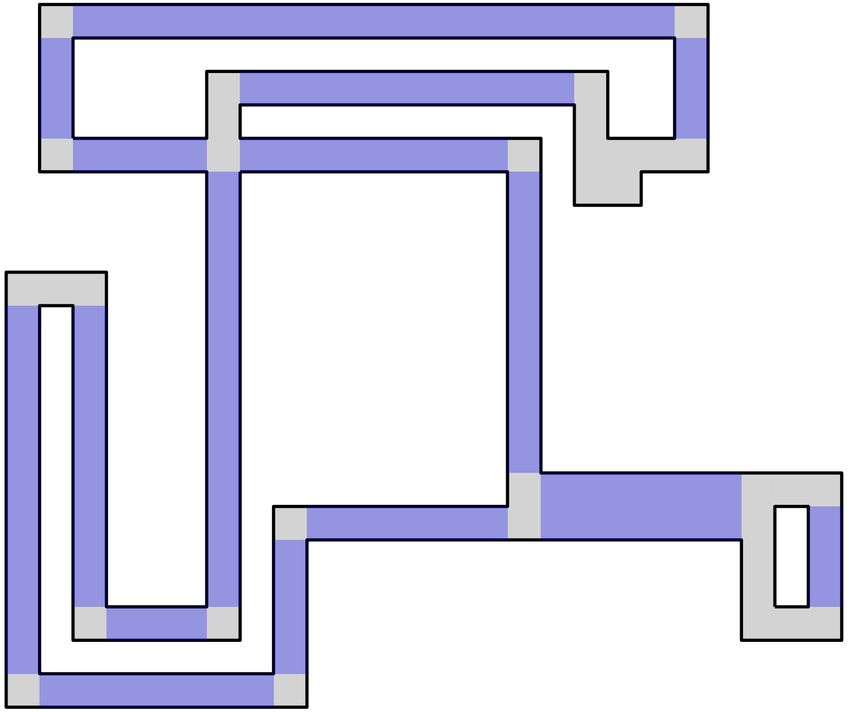
The final reduction



Find all pipes of length at least twice their width.

The final reduction

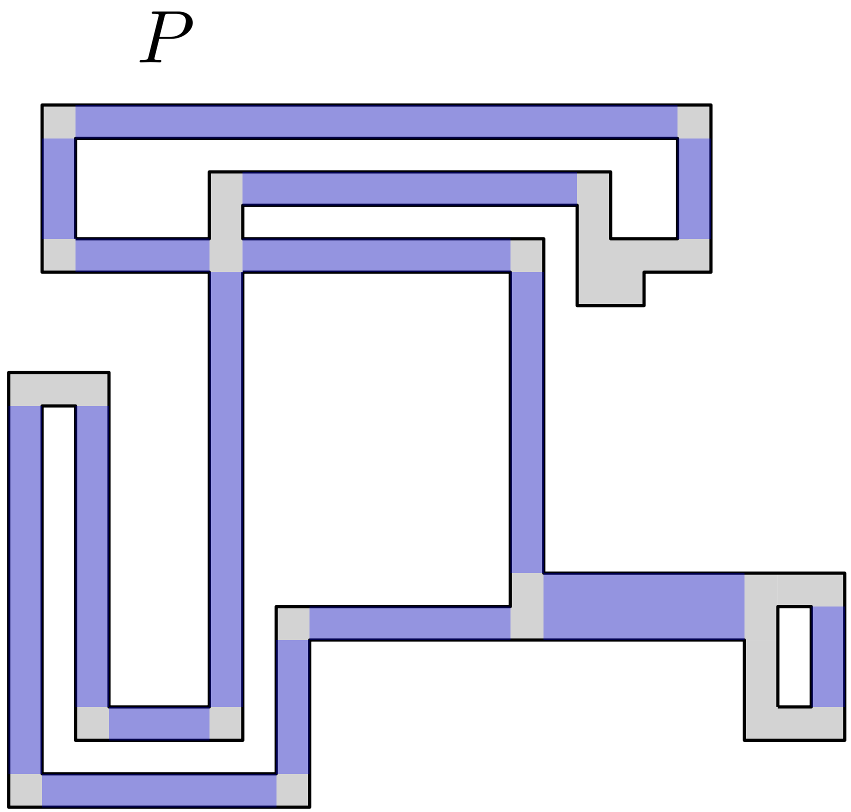
P



Find all pipes of length at least twice their width.

Perform the following operation on each pipe of $G(P')$

The final reduction



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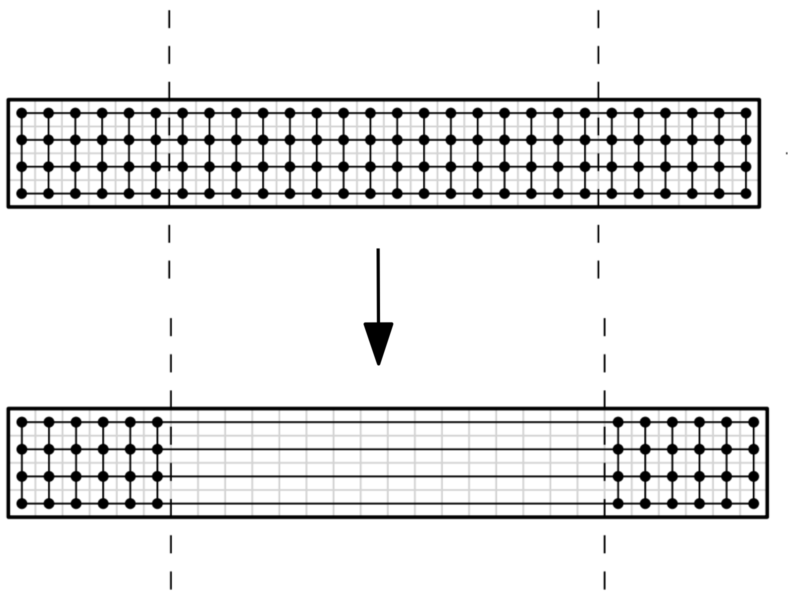
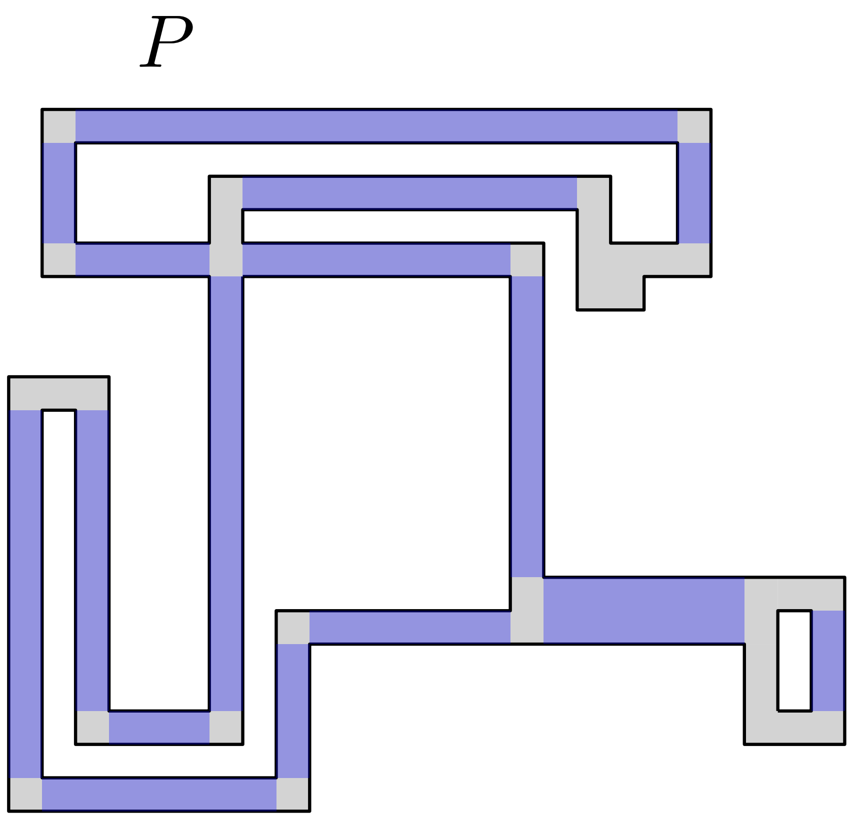


The final reduction



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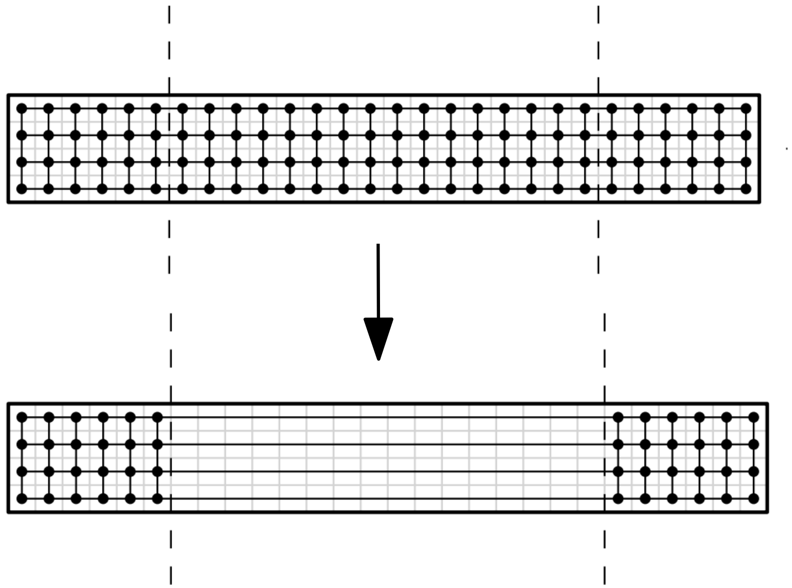
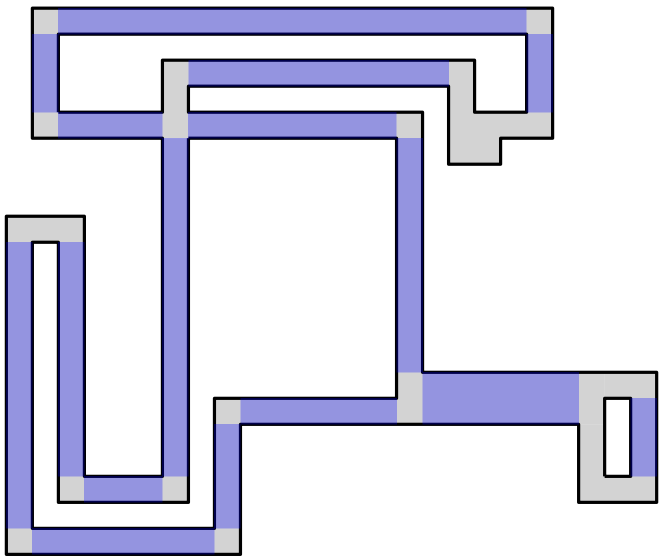
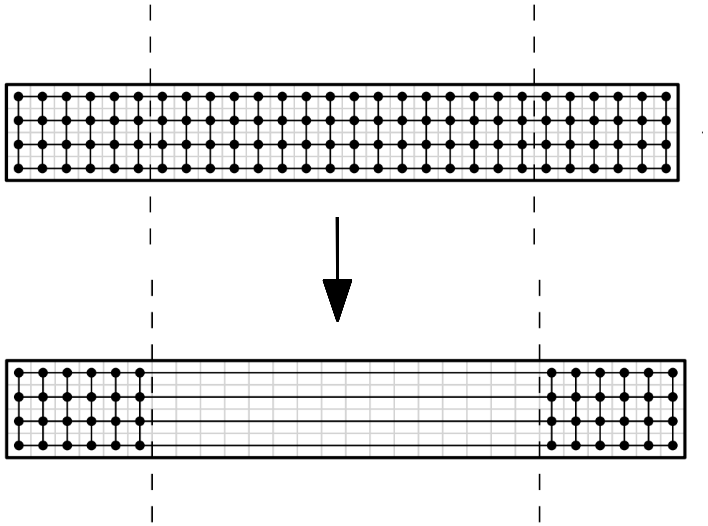


Lemma ensures that no. of unmatched vertices in a maximum matching remains the same

Summing up the reduction





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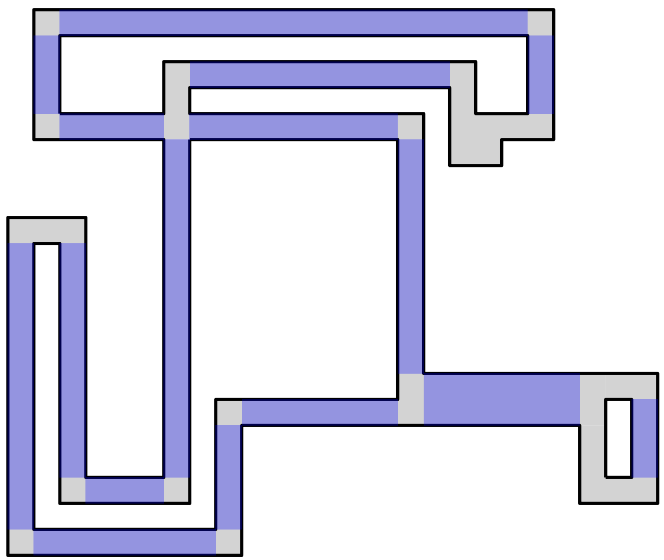
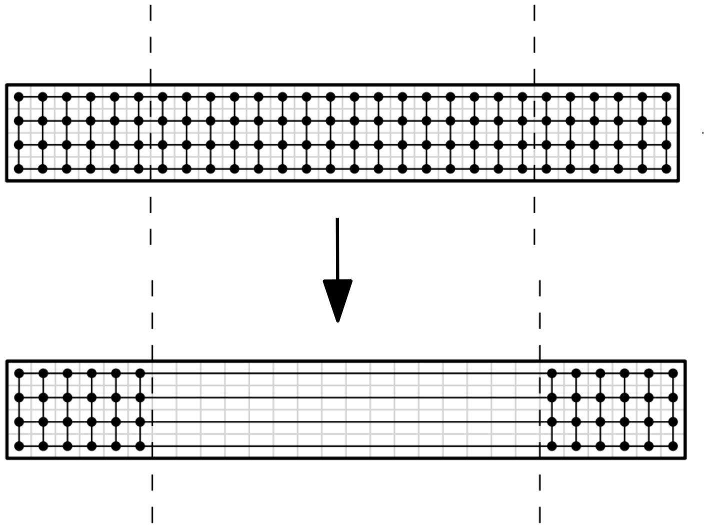
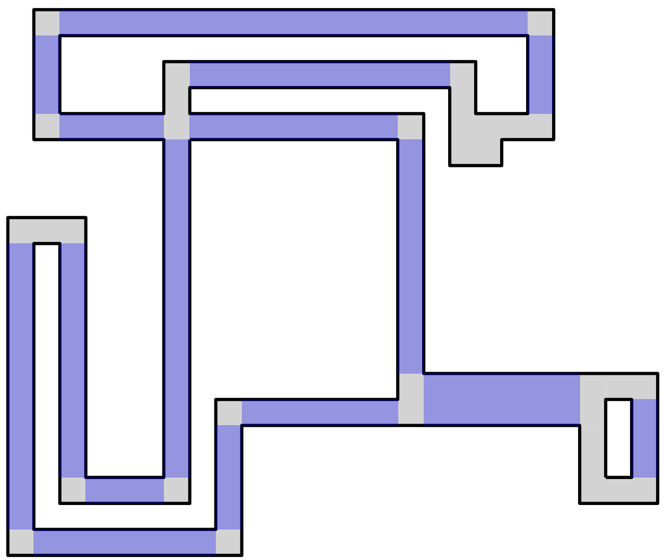
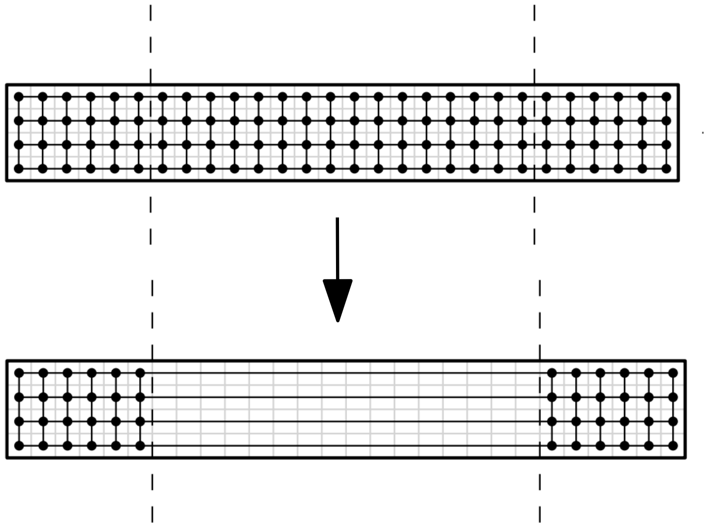




In reduced instance G^*, each vertex is of distance $O(n)$ to a corner.

Summing up the reduction





In reduced instance G^*, each vertex is of distance $O(n)$ to a corner.

Thus, G^* has order $O(n^3)$

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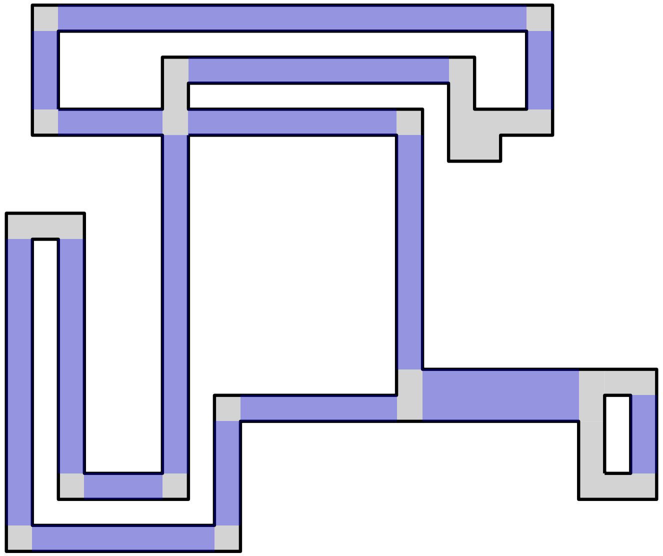
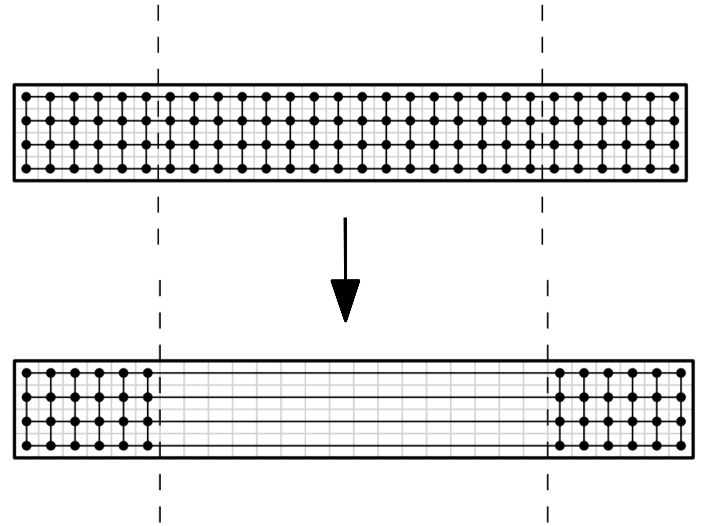


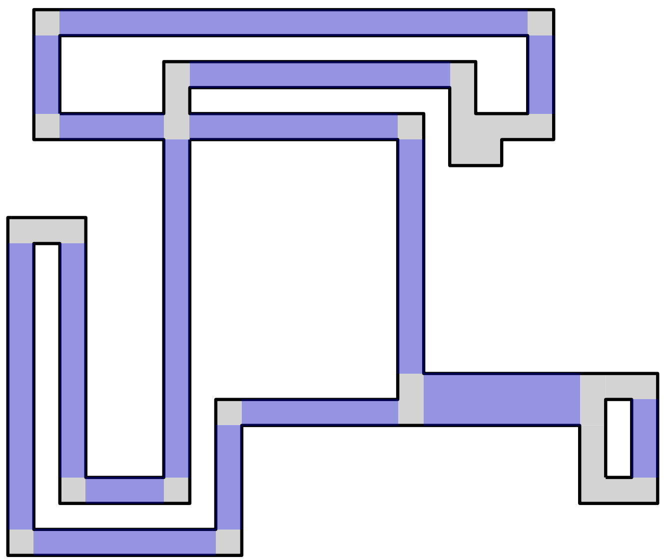


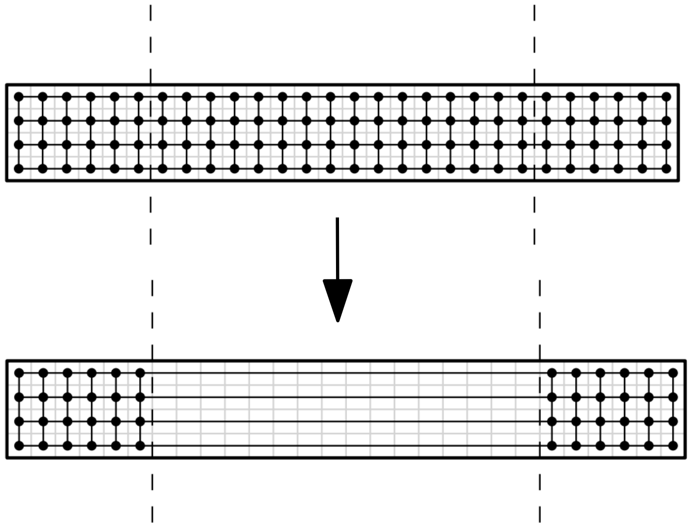
In reduced instance G^*, each vertex is of distance $O(n)$ to a corner.

Thus, G^* has order $O(n^3)$

G^* is planar and bipartite.

Summing up the reduction





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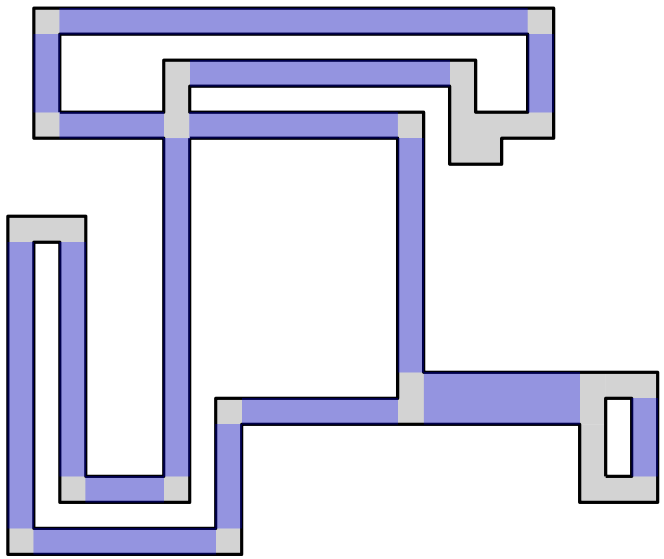
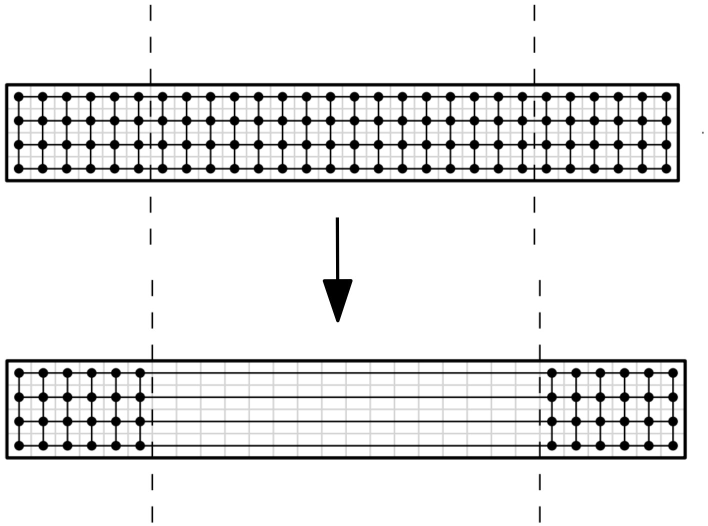
Find maximum matching M using a multiple-source multiple-sink maximum flow alg., $O(n^3 \log^3 n)$ time.

Summing up the reduction





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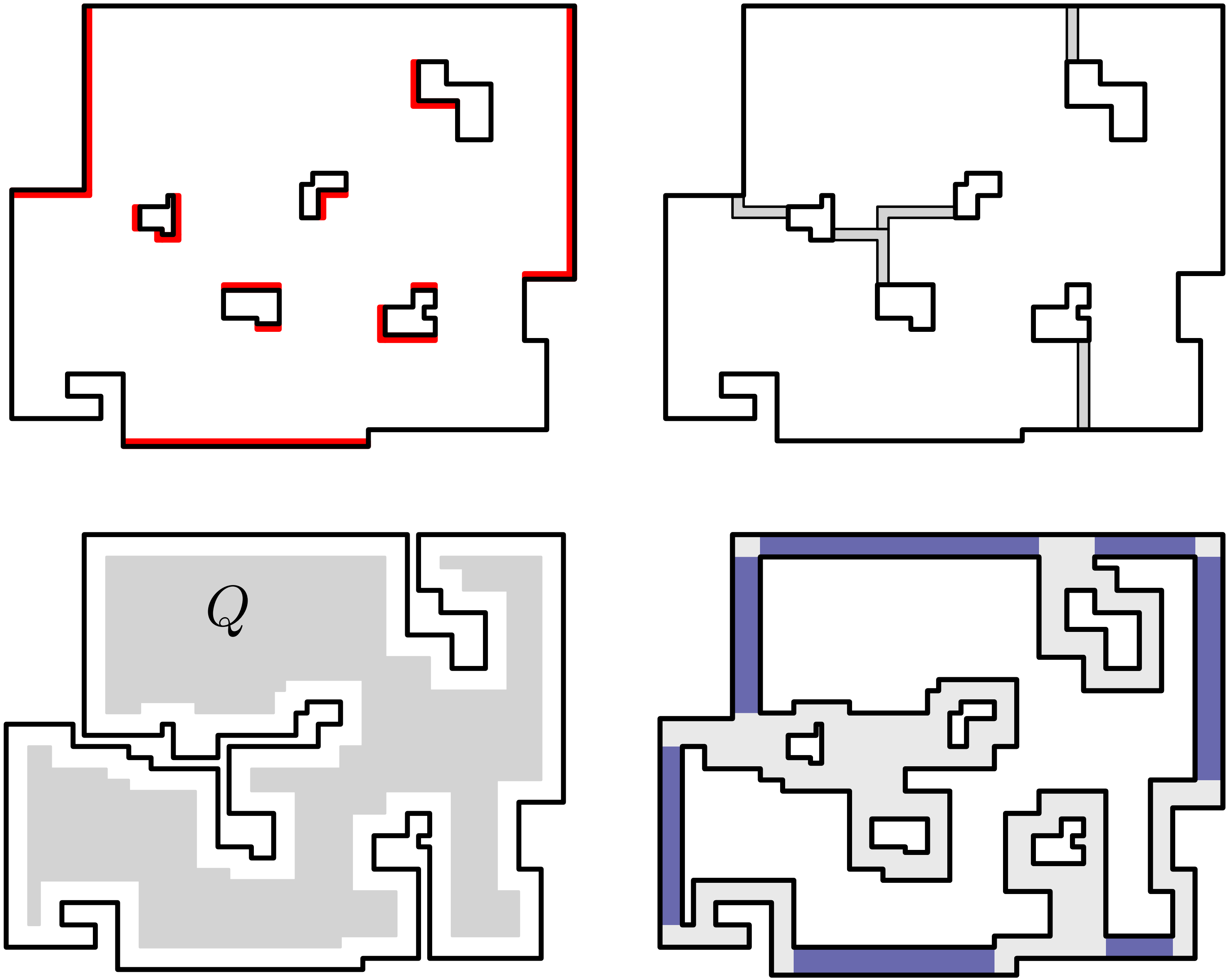
Find maximum matching M using a multiple-source multiple-sink maximum flow alg., $O(n^3 \log^3 n)$ time.

Return $|M| + \frac{\text{area}(P) - V(G^*)}{2}$.

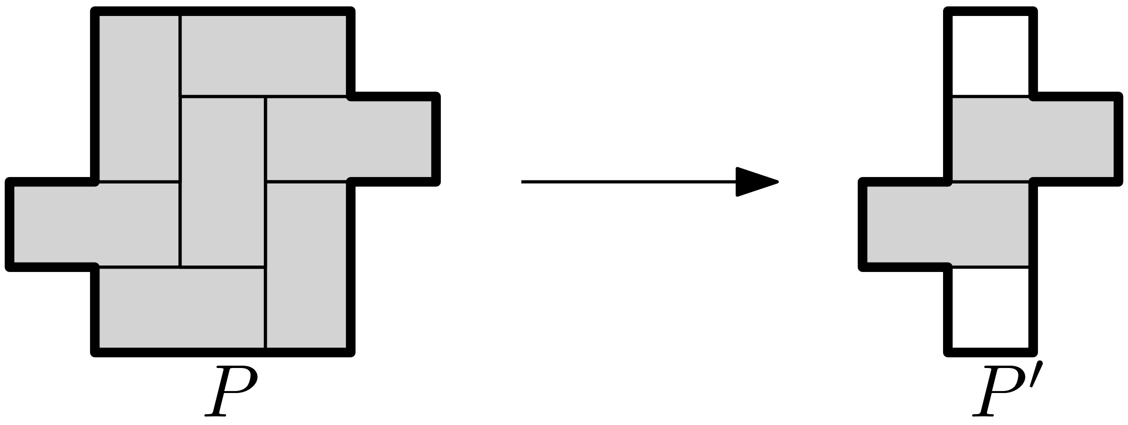
The total running time

	Running time:
Compute P_0	$O(n \log n)$
Compute offset	$O(n \log n)$
Find long pipes	$O(n \log n)$
Find maximum matching	$O(n^3 \log^3 n)$

What if there are holes?



Simple algorithm



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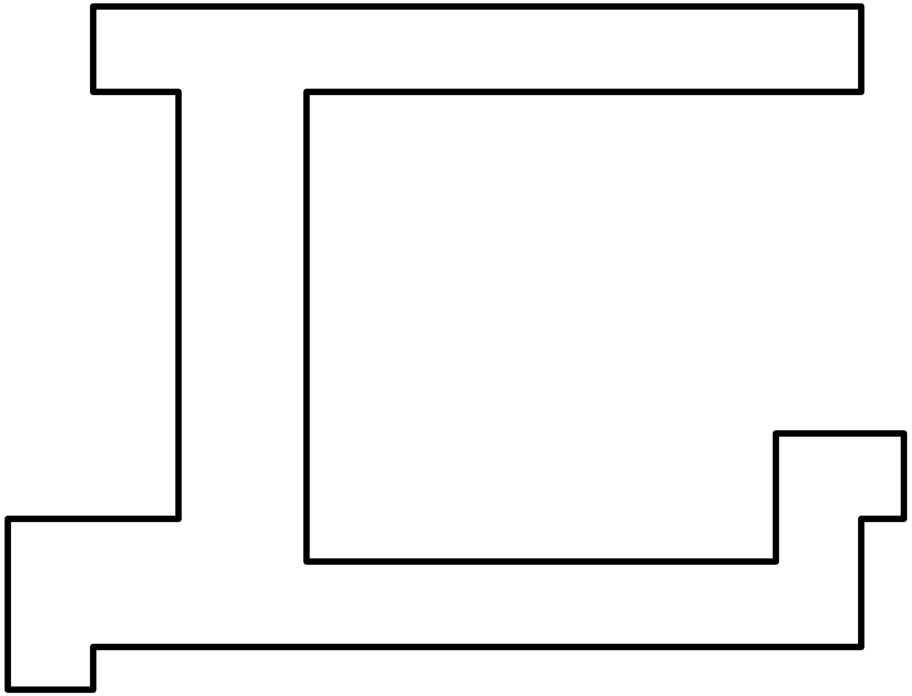

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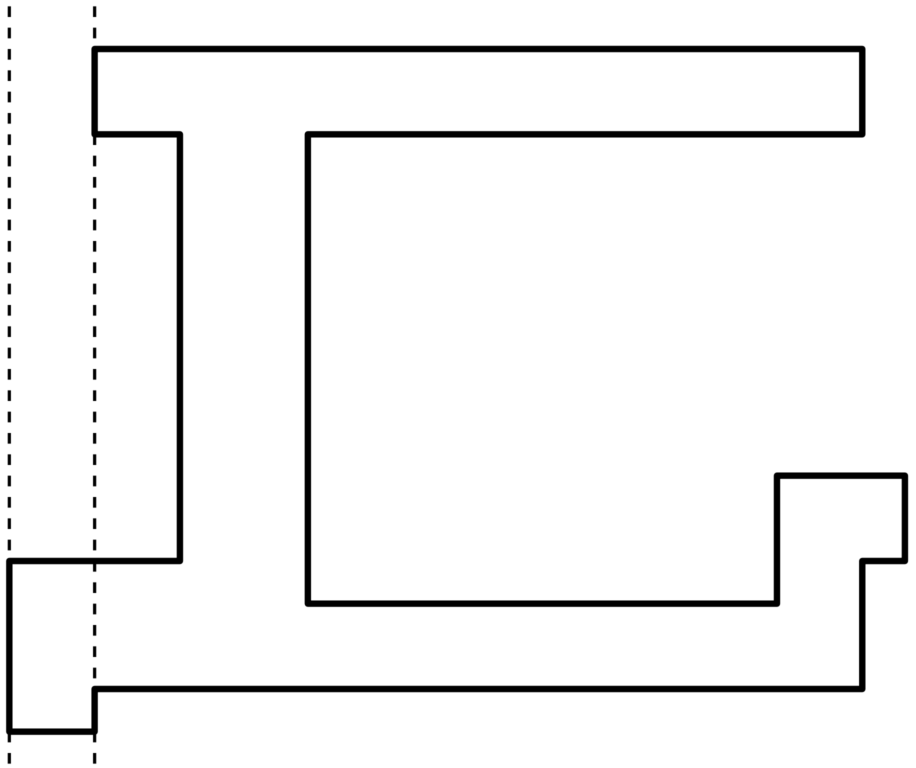

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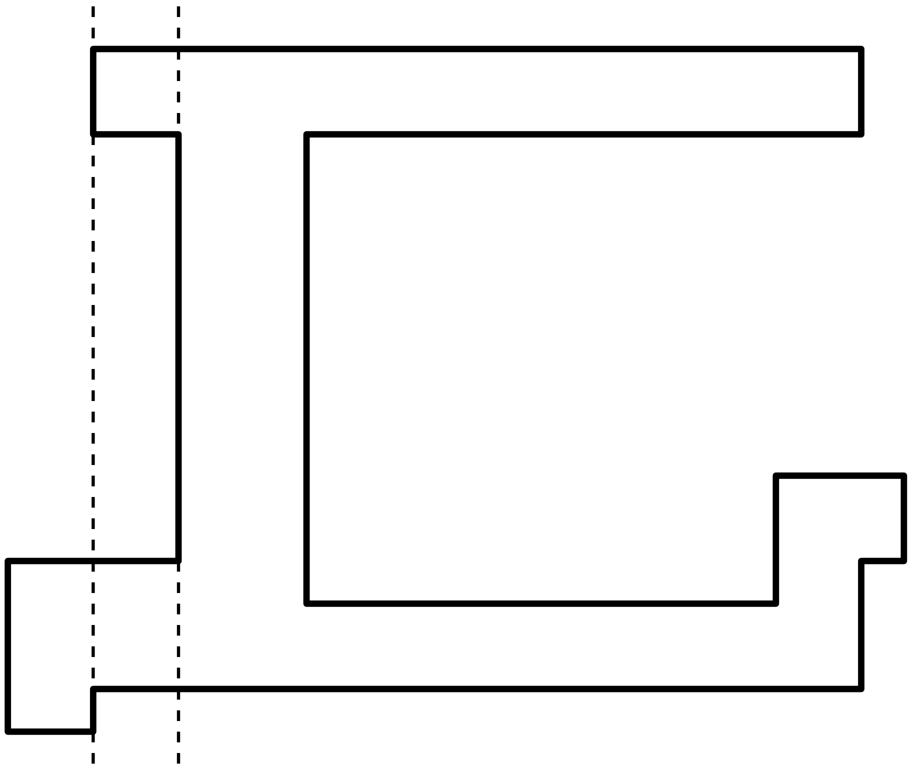

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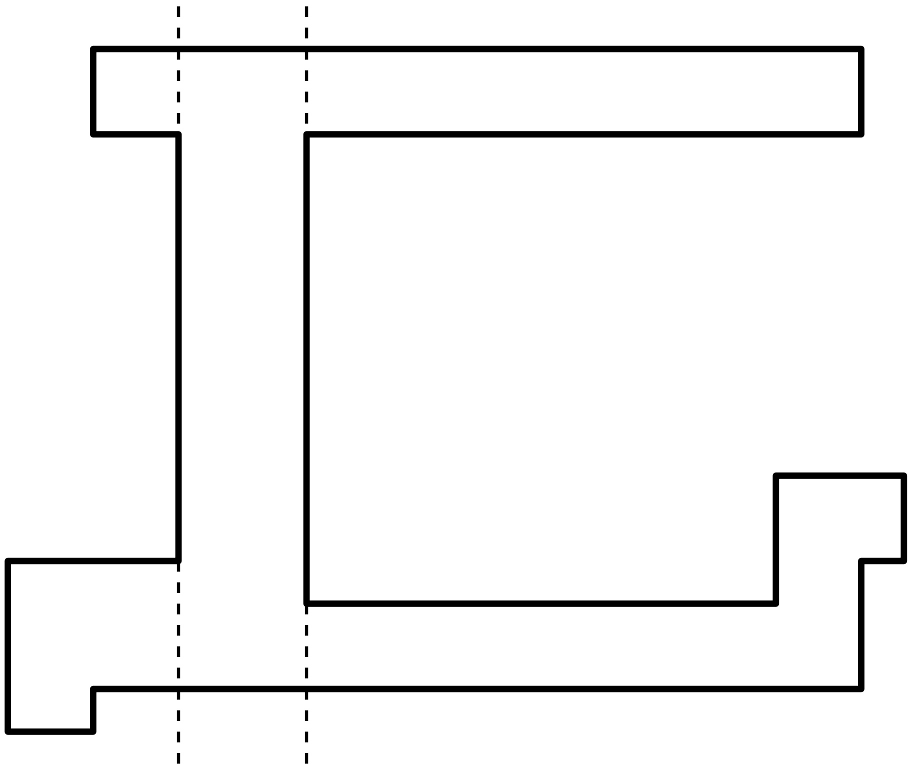

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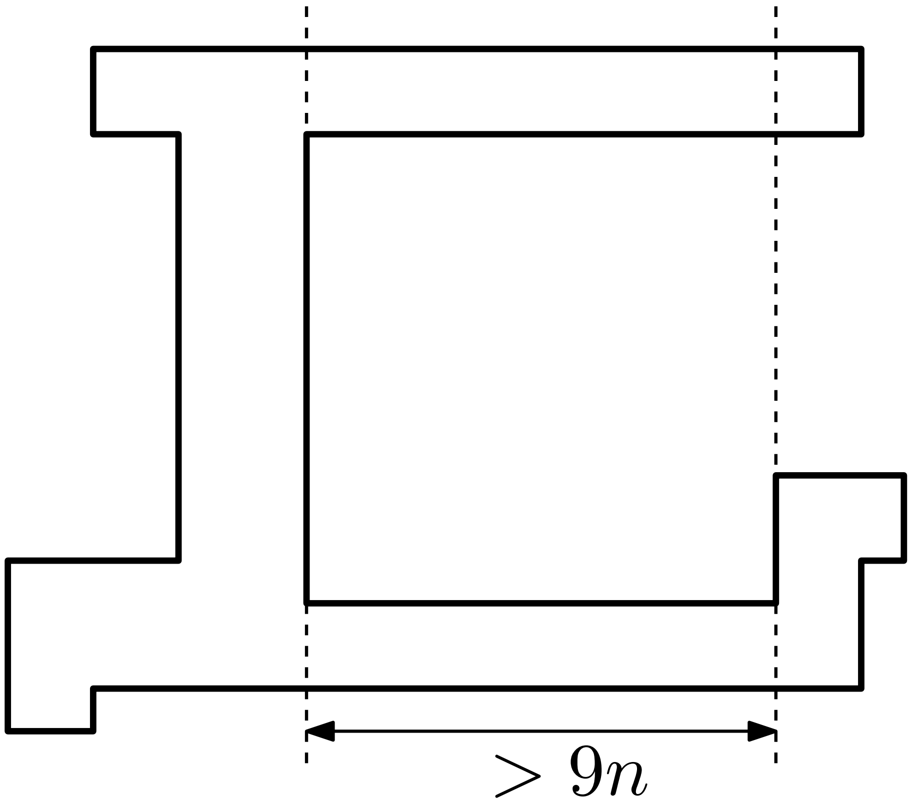

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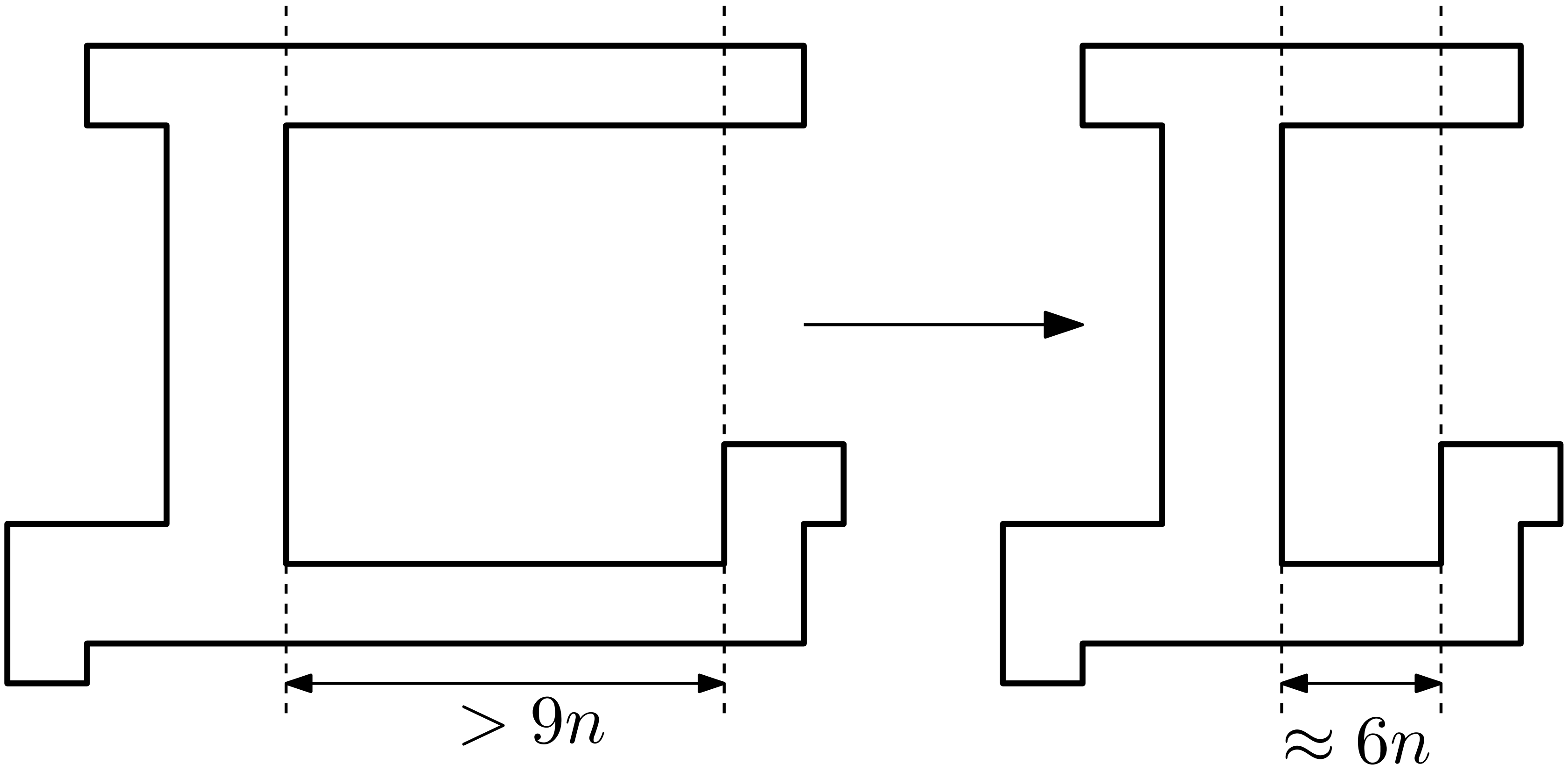

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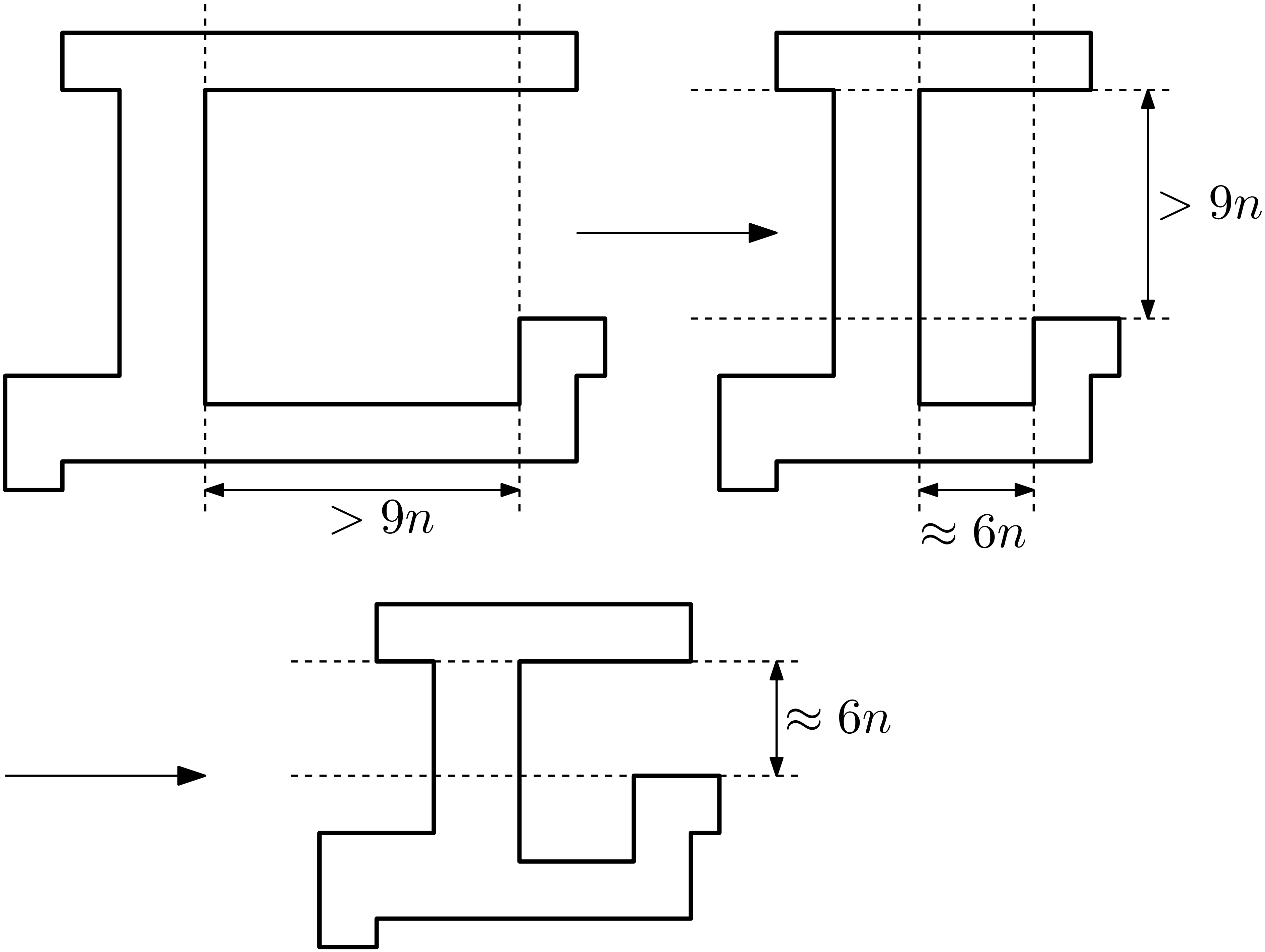
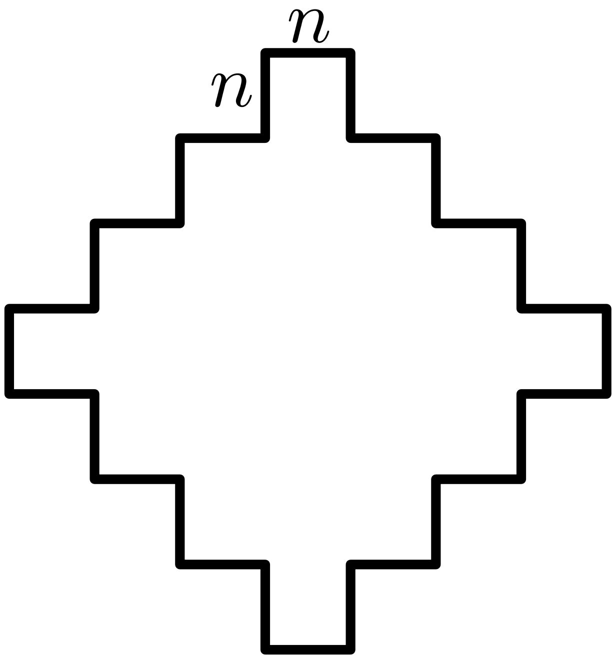
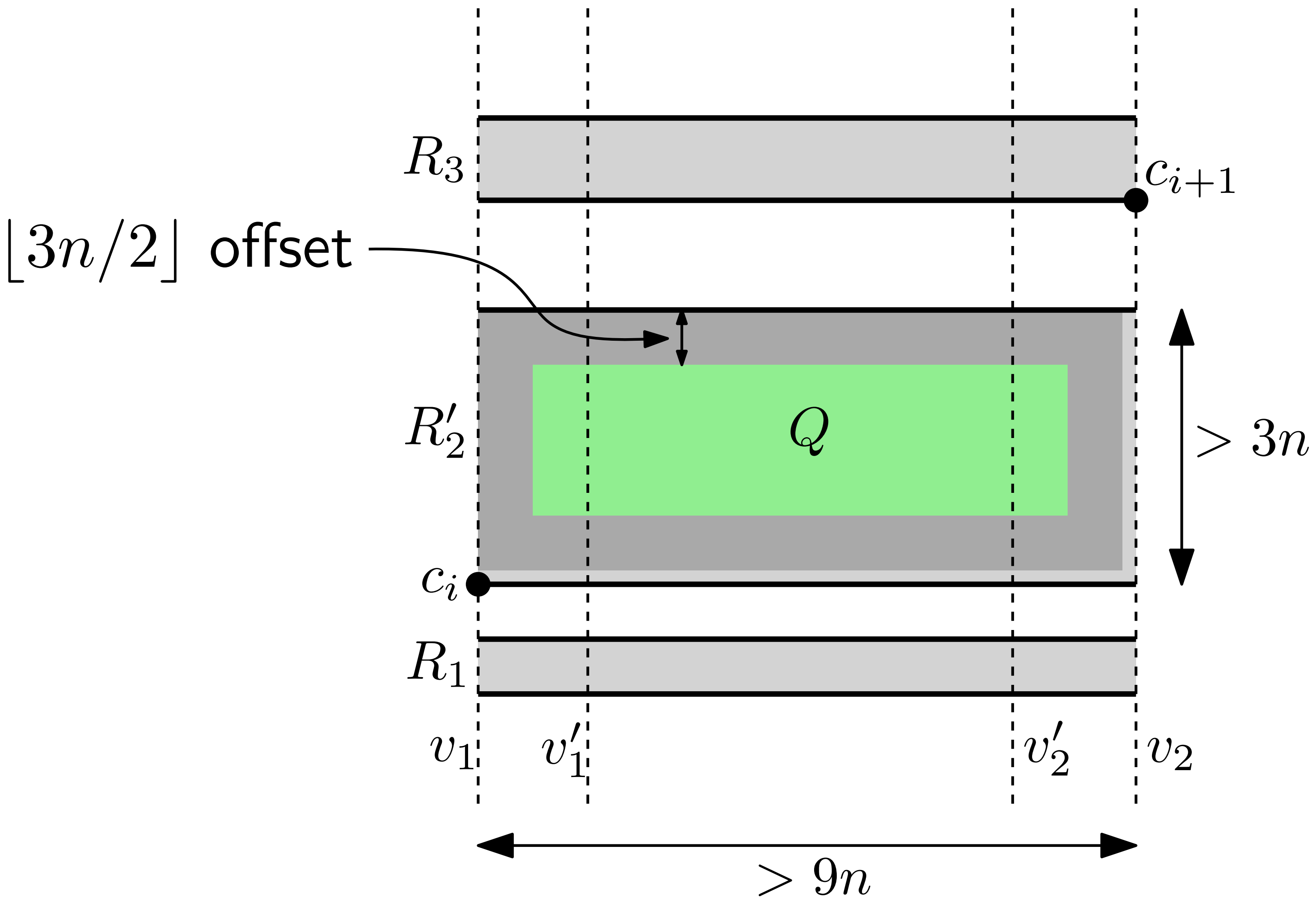


Running time



$$\widetilde{O}(n^4)$$

Correctness of simple algorithm



Open Problems

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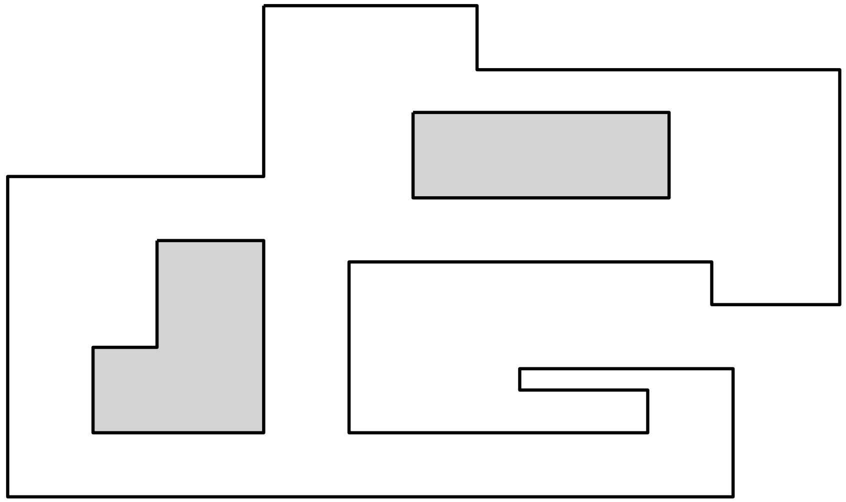
Can domino tiling/packing be solved faster with a reduction to a flow problem?



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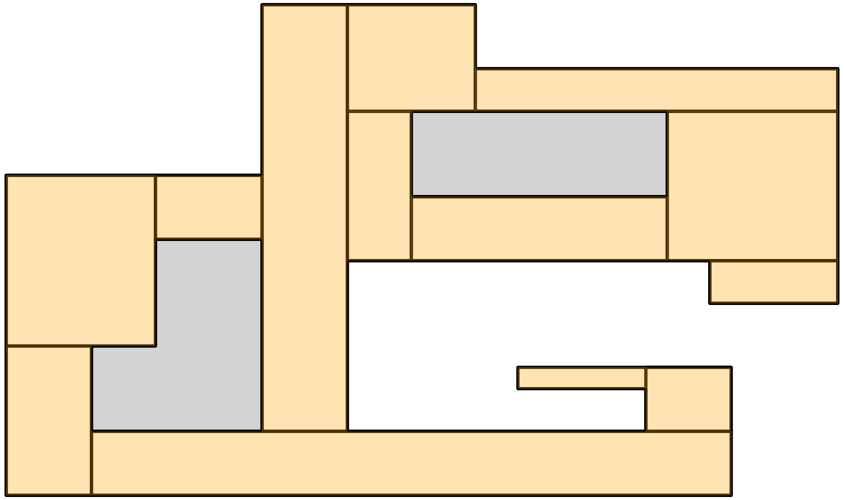

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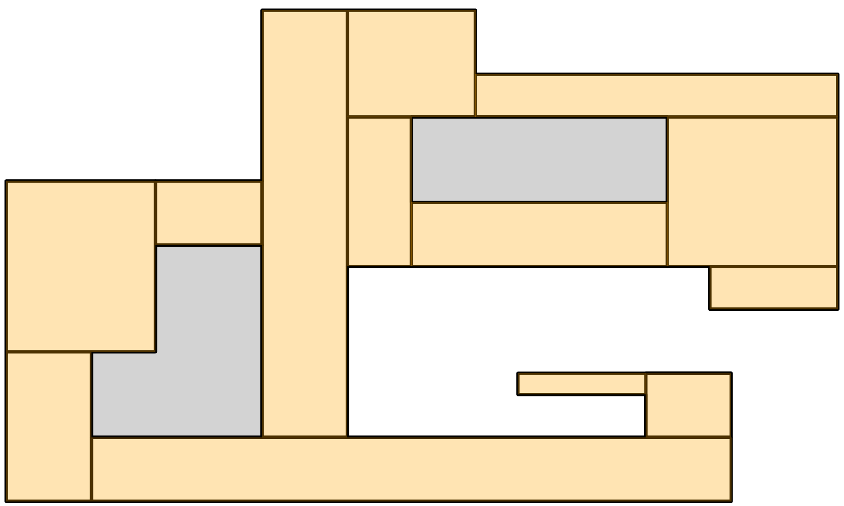

Packing 2×2 squares is NP-complete when P has holes. Can it be solved in polynomial time if P is hole-free?