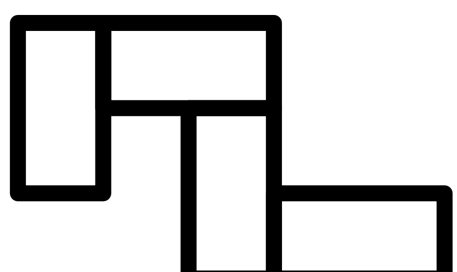
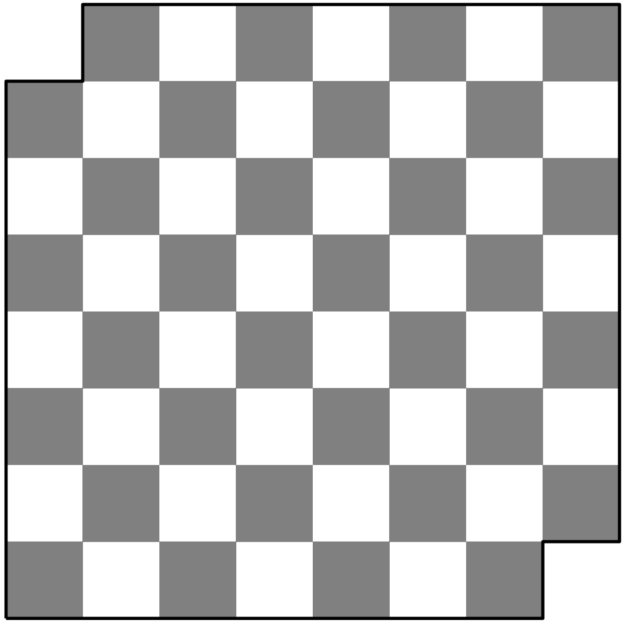
Tiling with Squares and Packing Dominos in Polynomial Time

Anders Aamand, Mikkel Abrahamsen, Thomas D. Ahle, Peter M. R. Rasmussen



Max Black, 1946: Two diagonally opposite corners have been removed from a chessboard. Can 31 1×2 dominos be placed to cover the remaining squares?



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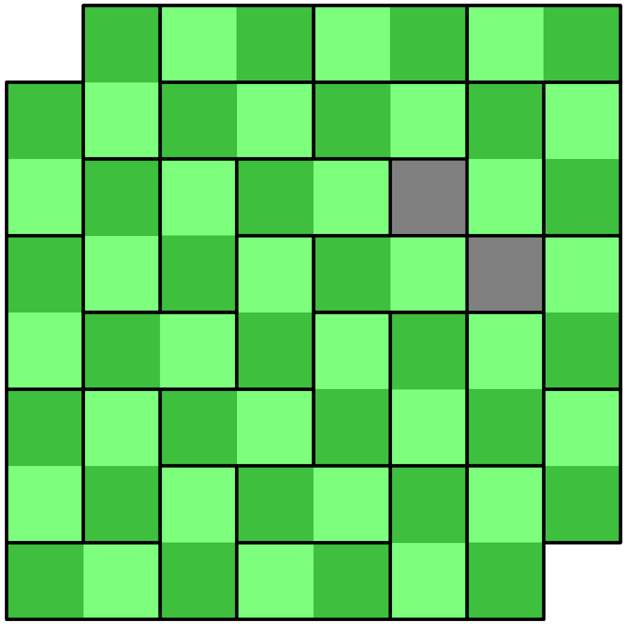
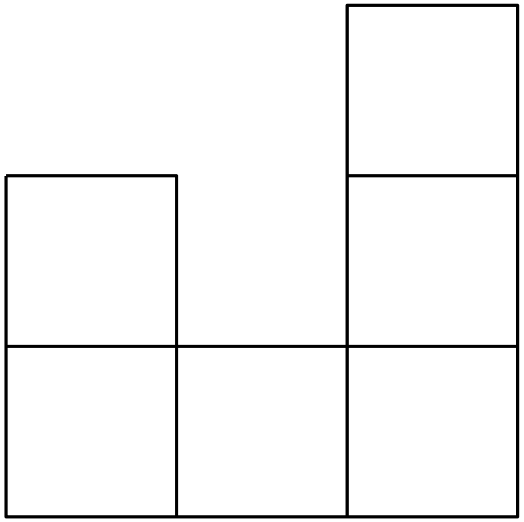


International Mathematical Olympiad 2004:
For which m and n can an $m \times n$ rectangle be tiled with 'hooks' of the following type:



Motivation

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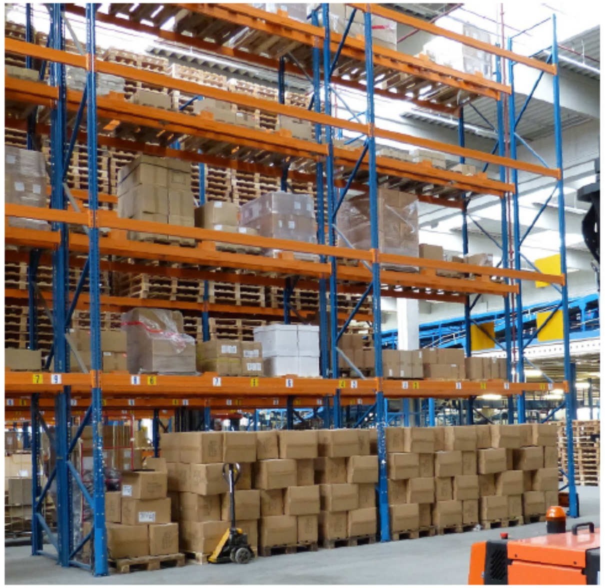

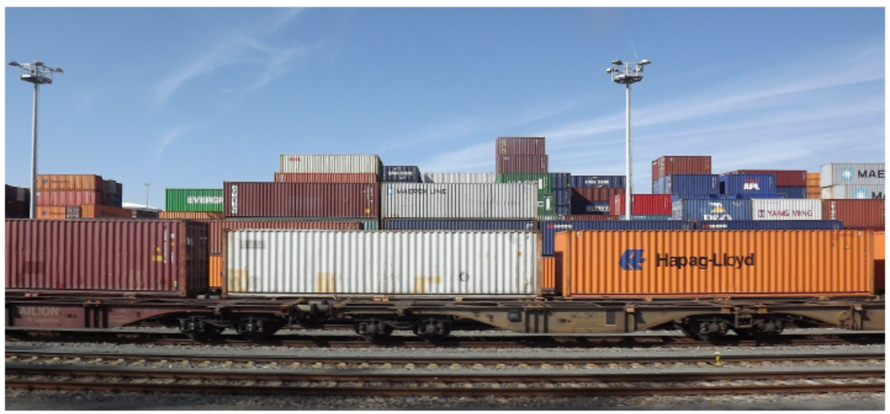

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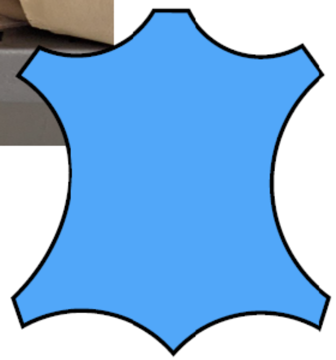
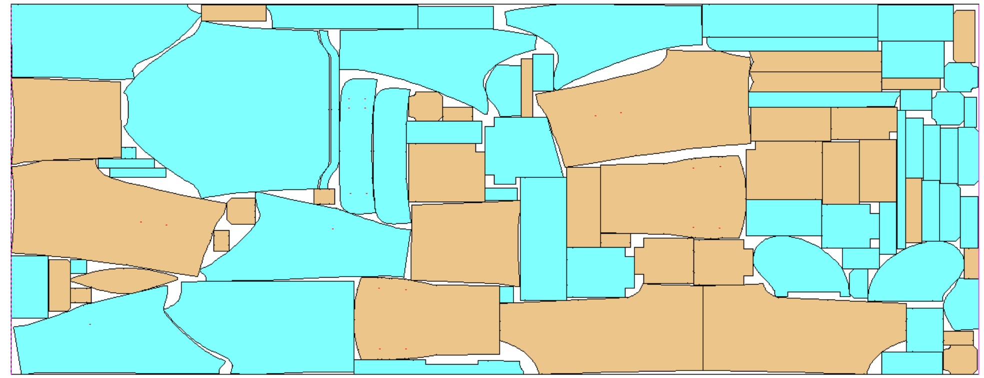
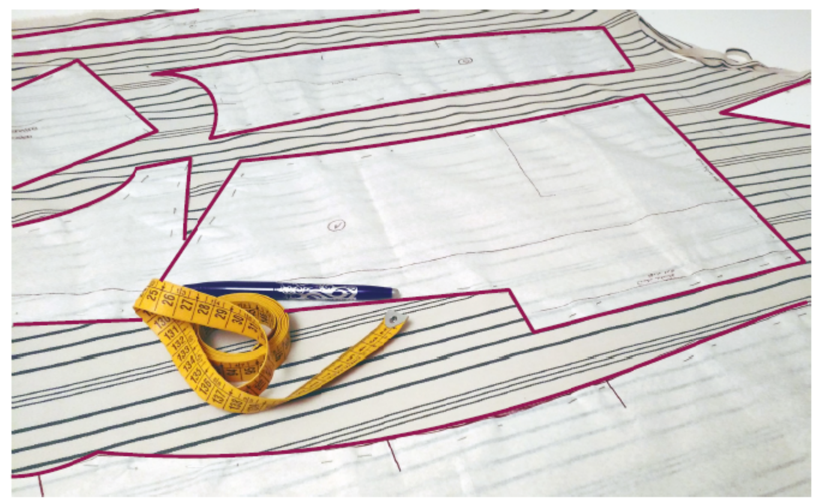
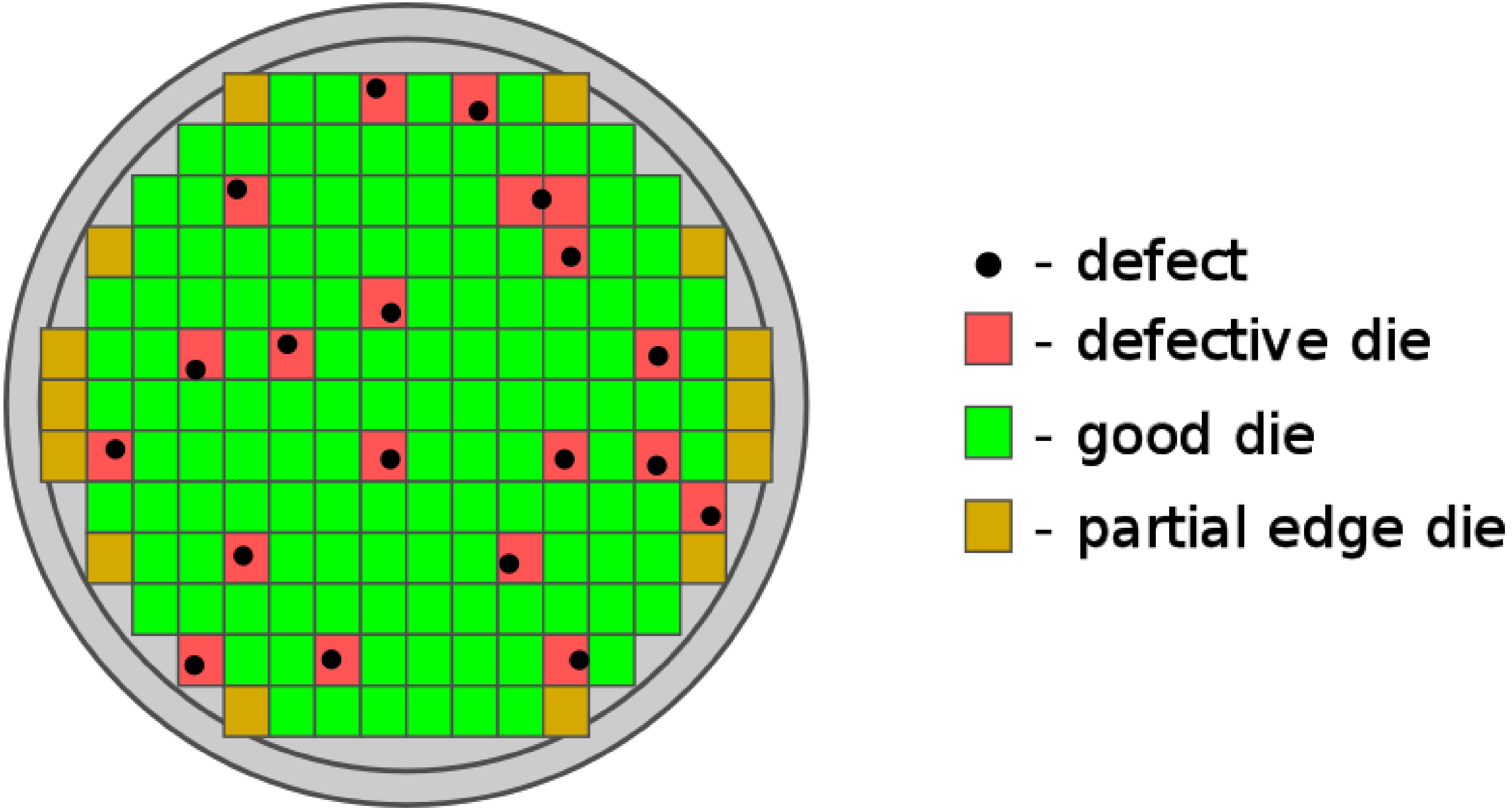






Motivation of domino packing



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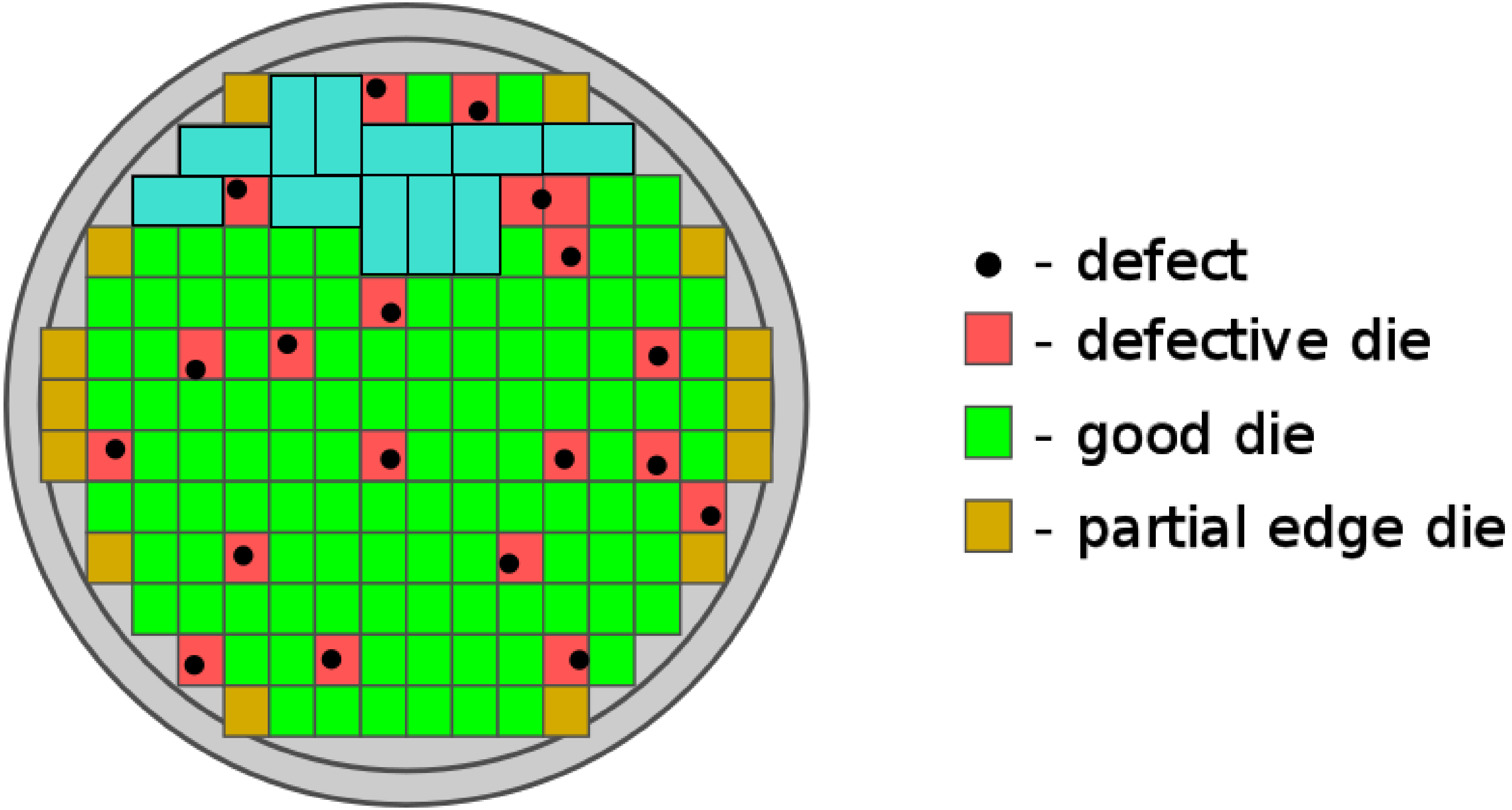
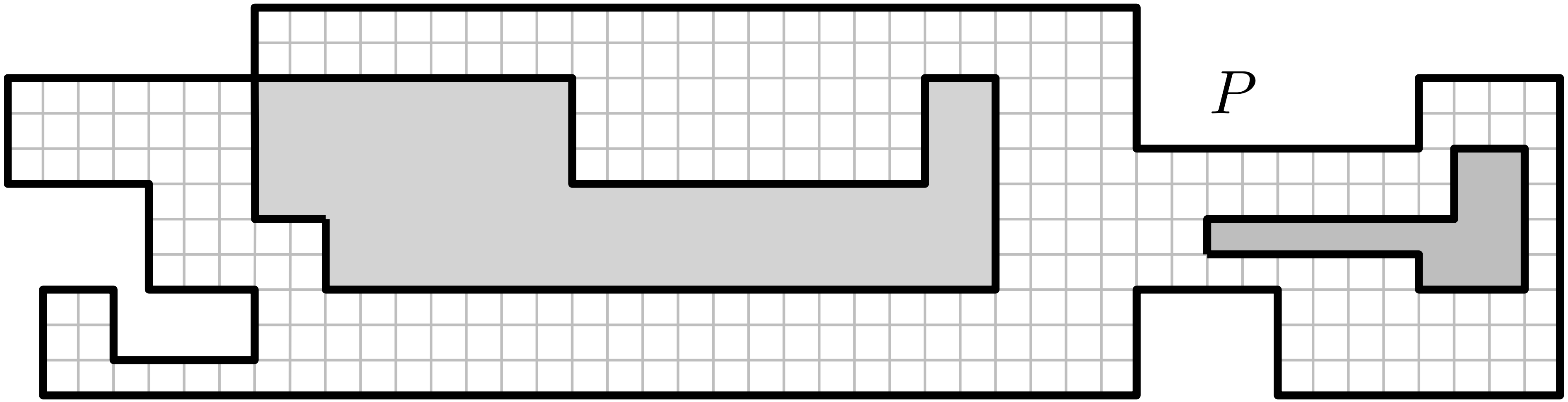


Polyomino: A polygonal region in the plane with axis-parallel edges and corners of integral coordinates.



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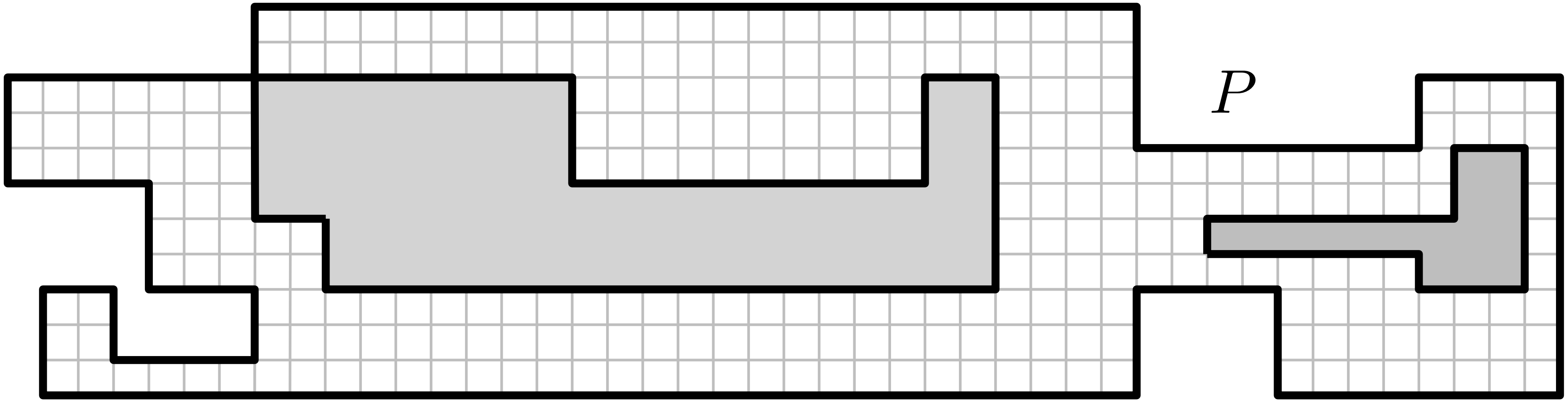
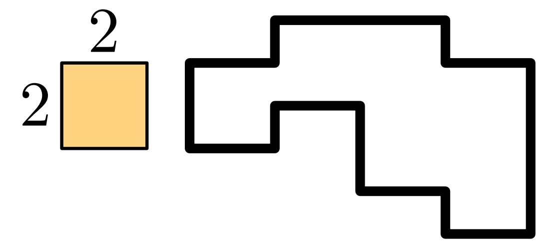


Tiling: Can a given large polyomino P be tiled with copies of a given small polyomino Q?



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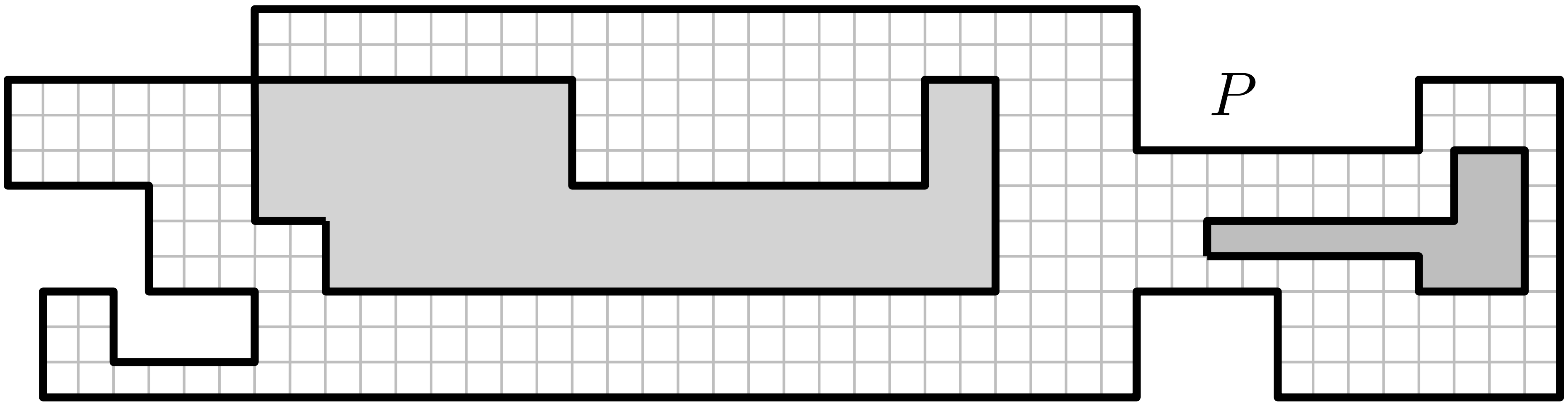

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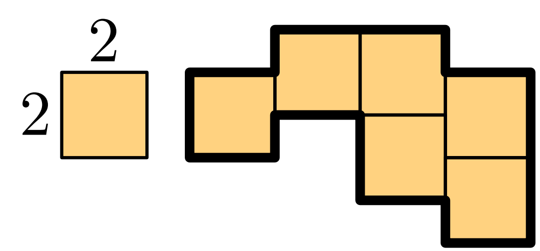

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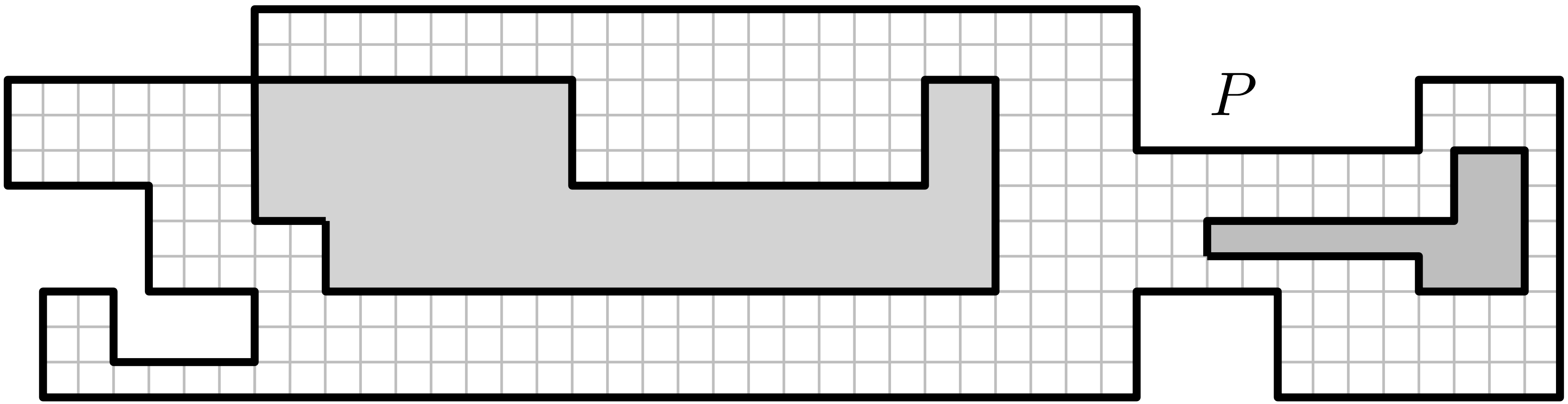

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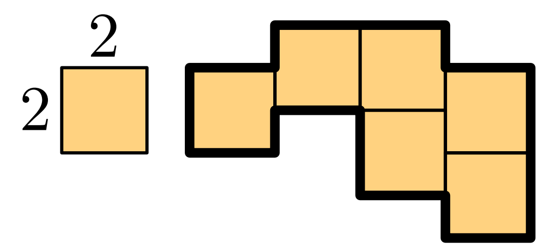
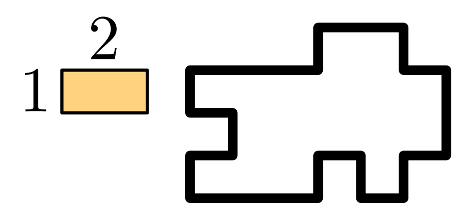


Packing: How many non-overlapping copies of Q can be fit inside P?



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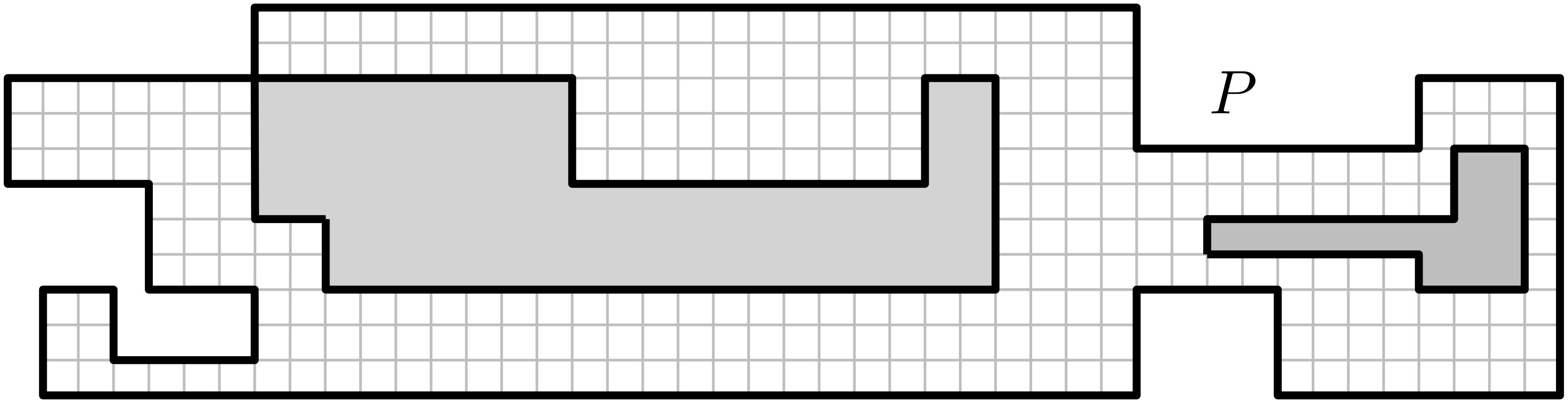

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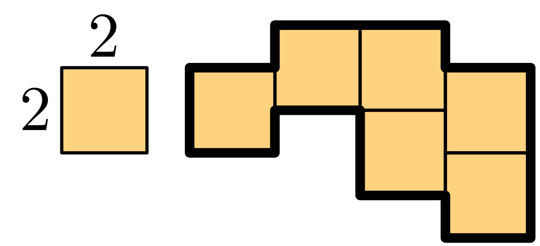

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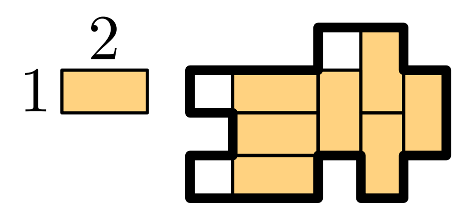

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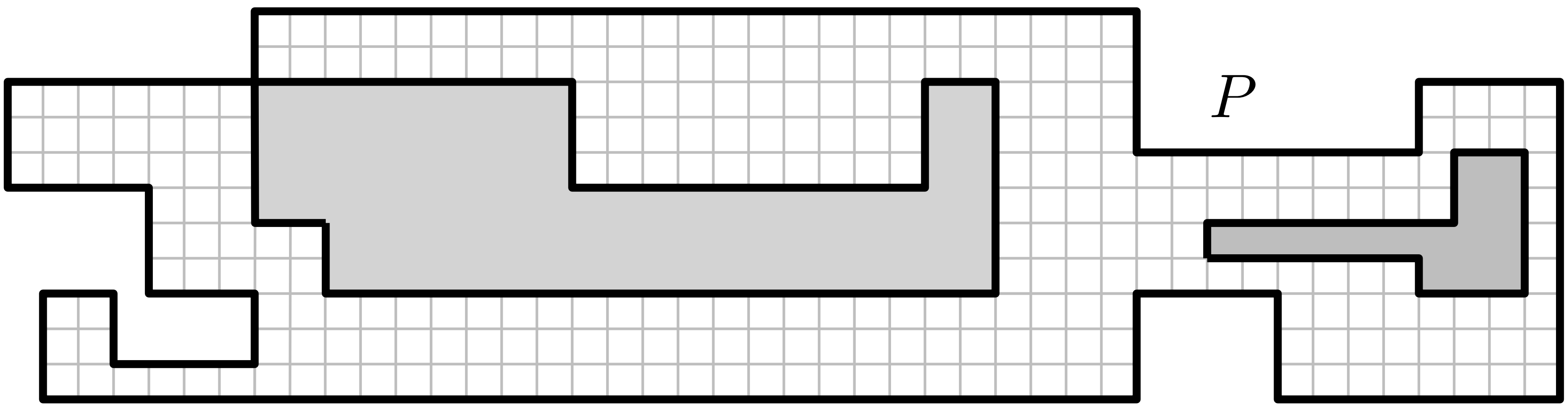

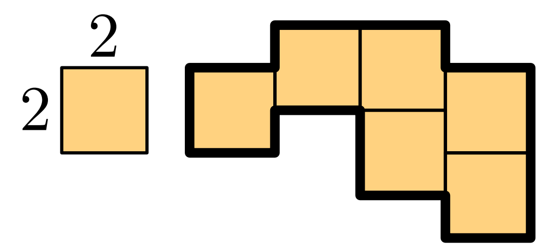

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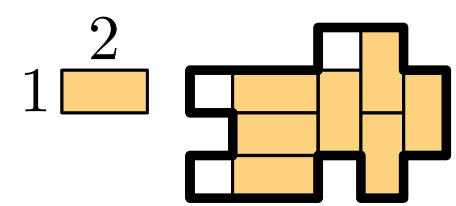
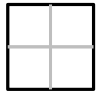
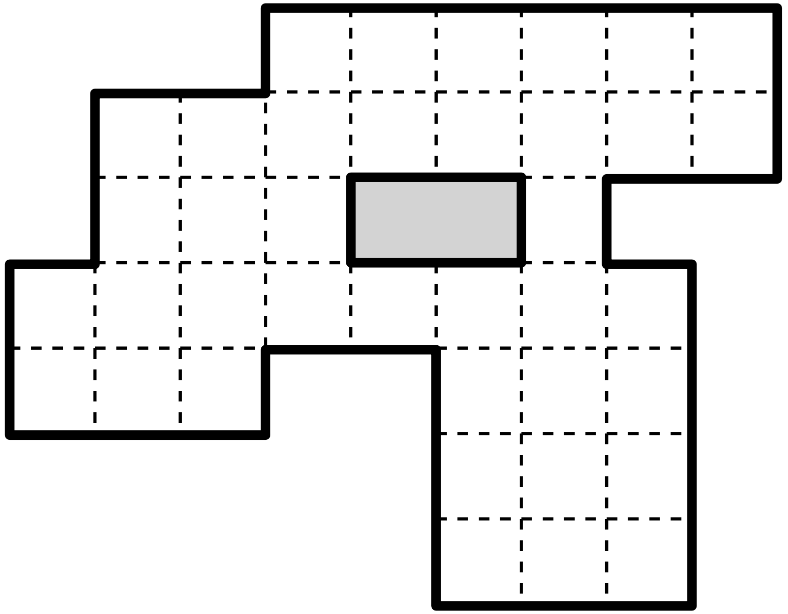


Our paper: $Q \in \{$  ,  $\}$

Representing a polyomino



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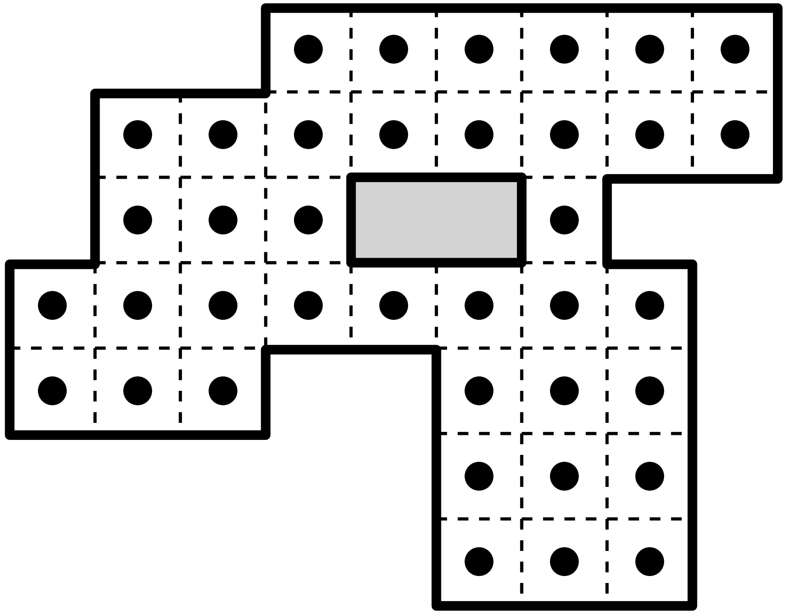


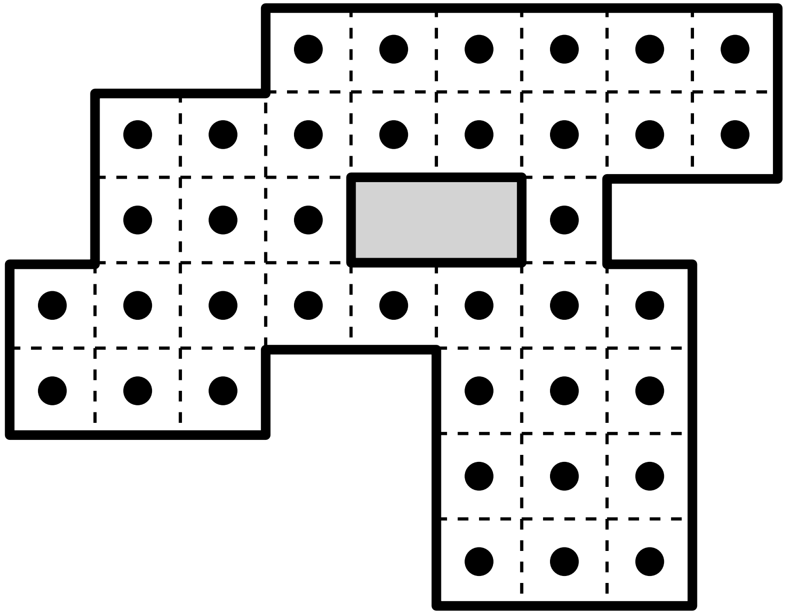
Usual way:

Store coordinates of each cell:

[• , • , • , • , • , • , . . .]

Area representation

Representing a polyomino



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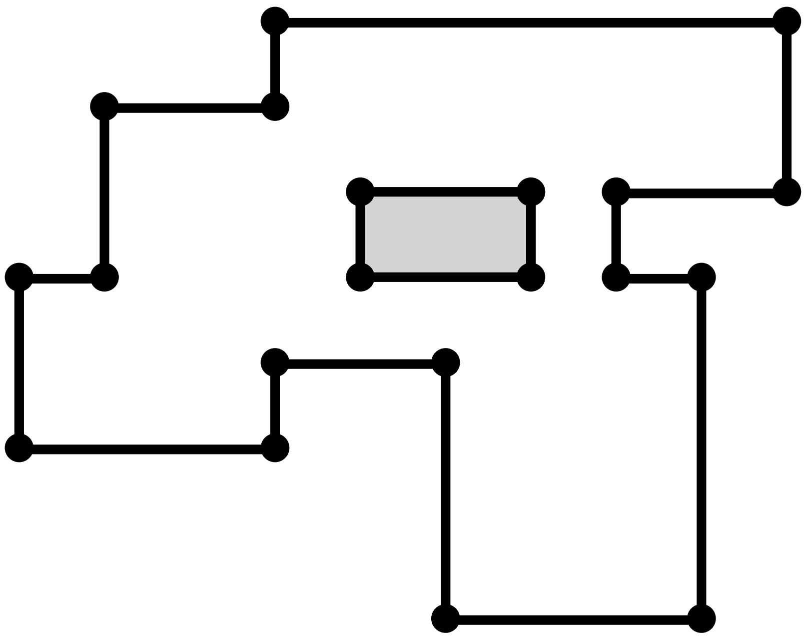
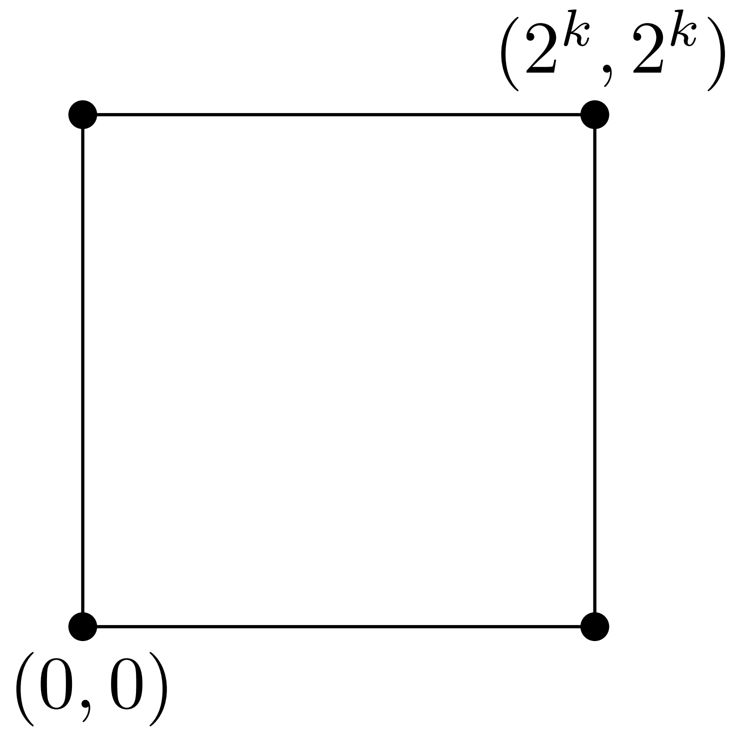


Compact way:

Store coordinates of corners.

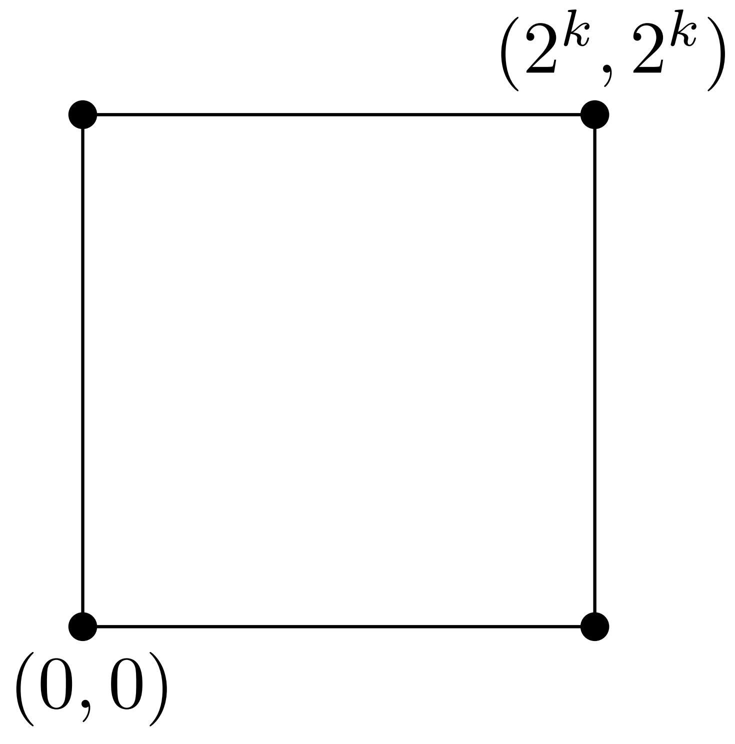
Corner representation

Example



Example

Corner representation:
$[(0,0),(2^k,0),(2^k,2^k),(0,2^k)]$



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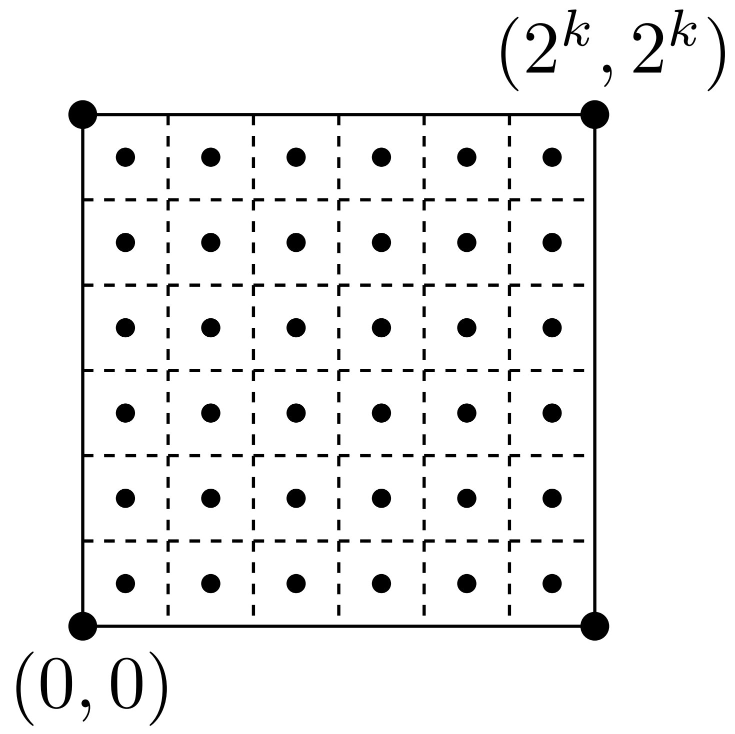
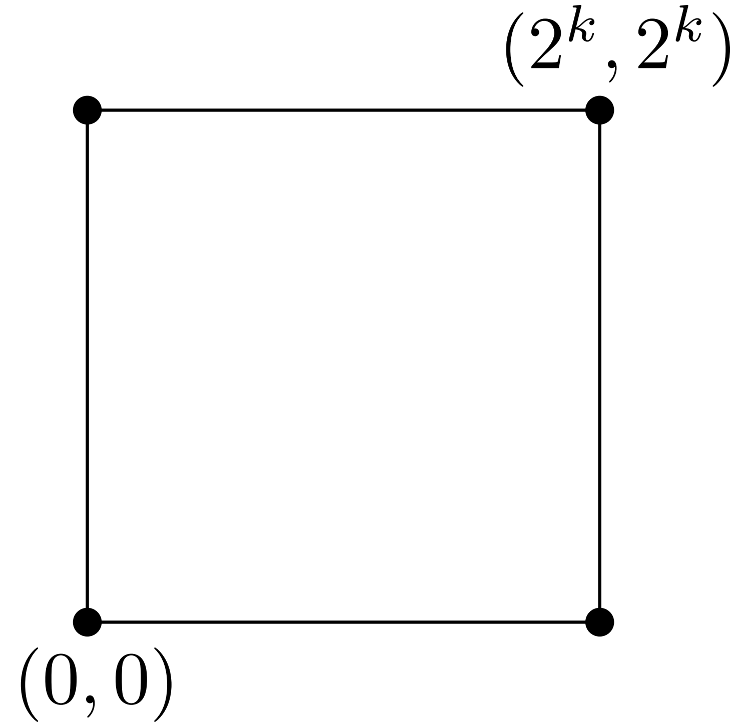


Area representation:
$[(0,0),(1,0),(2,0),\ldots,(2^k,0),$
$(0,1),(1,1),(2,1),\ldots,(2^k,1),$
$\vdots$
$(0,2^k),(1,2^k),(2,2^k),\ldots,(2^k,2^k)]$

Goal

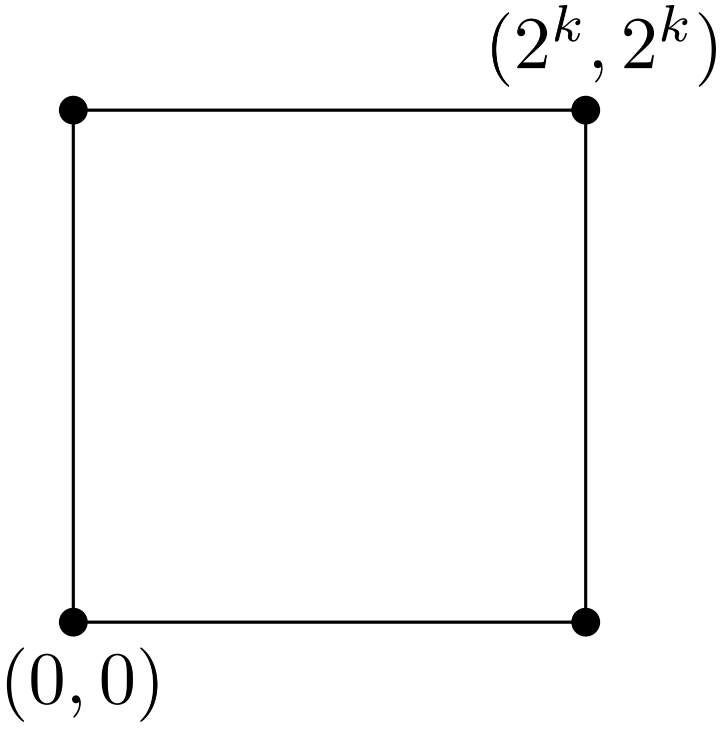
Known algorithms:
Assume area representation $\Rightarrow$
Time polynomial in the area.



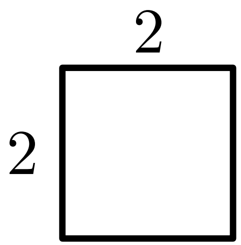
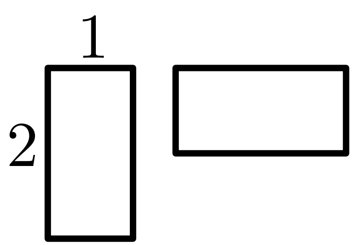
Goal

Known algorithms:
Assume area representation $\Rightarrow$
Time polynomial in the area.

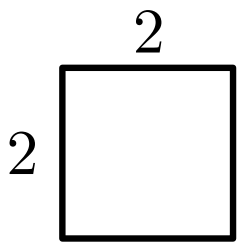
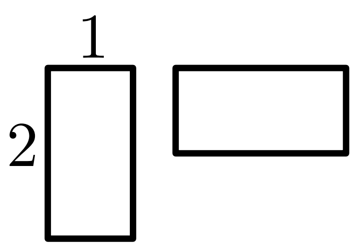
Goal:
Assume corner representation.
Find algorithms with running time $O(\text{poly}(n))$.
n: the number of corners.



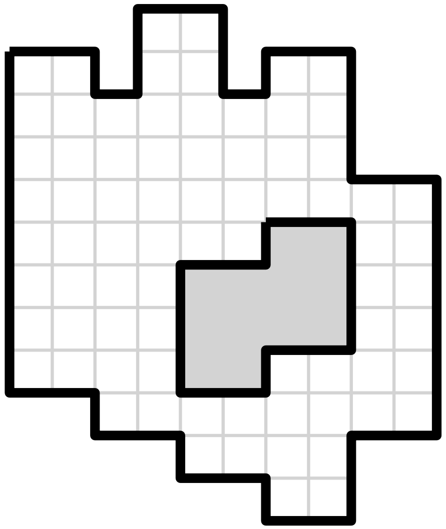
Results

Shapes	Tiling	Packing
	?	NP-complete
	?	?

Results

Shapes	Tiling	Packing
	No holes: $O(n)$ Holes: $O(n \log n)$	NP-complete
	$\widetilde{O}(n^3)$	$\widetilde{O}(n^3)$

Tiling with 2×2 squares



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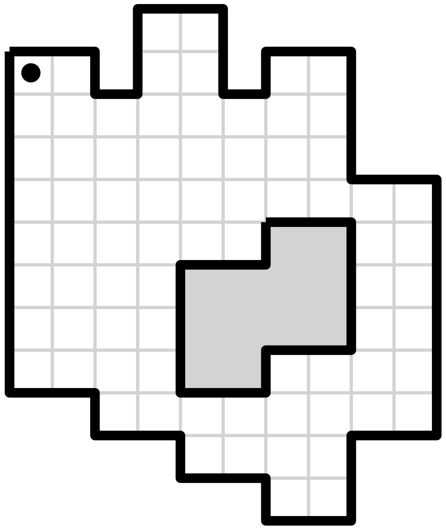

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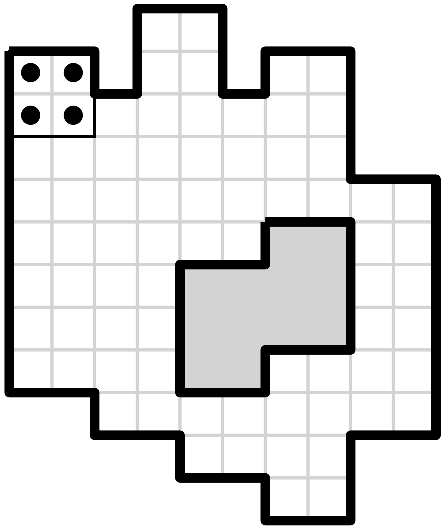

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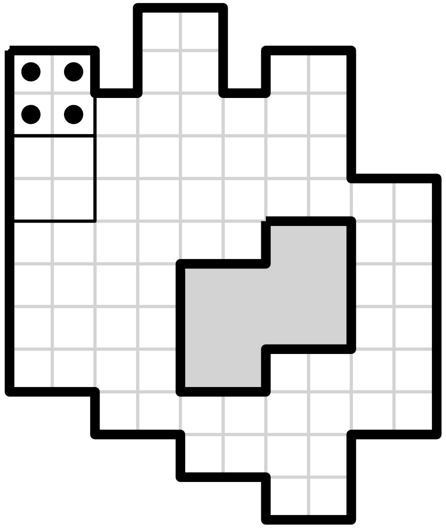

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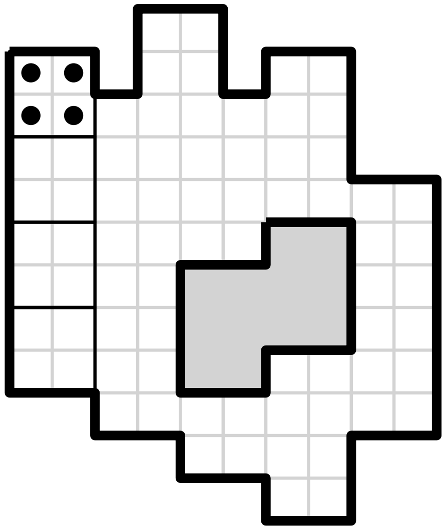

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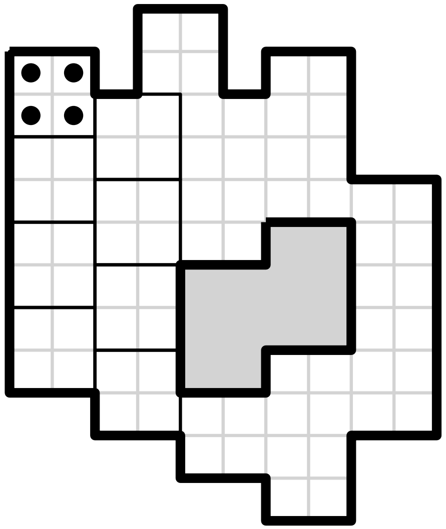

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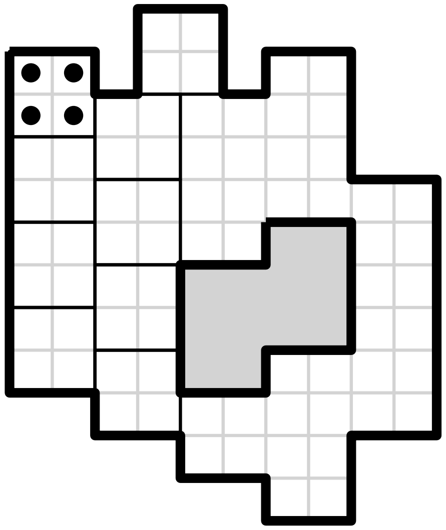

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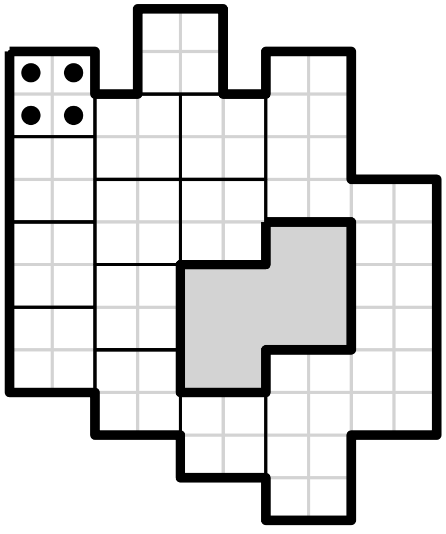

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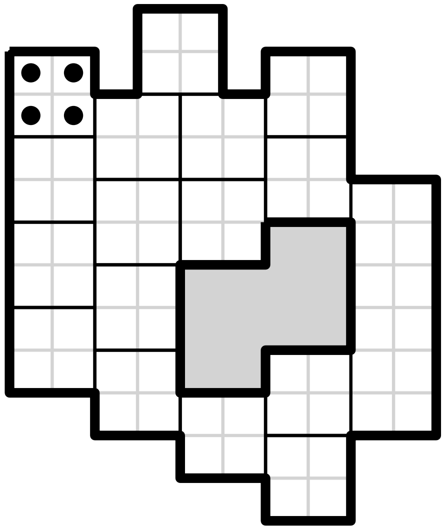

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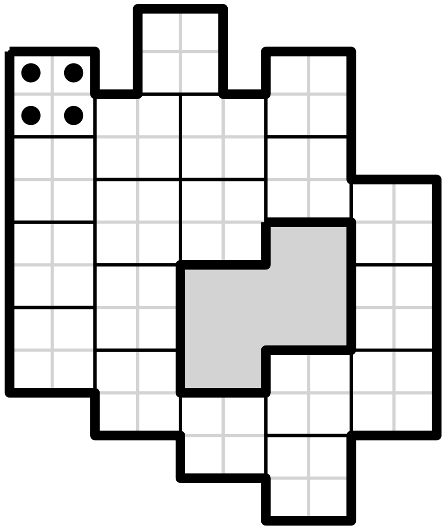

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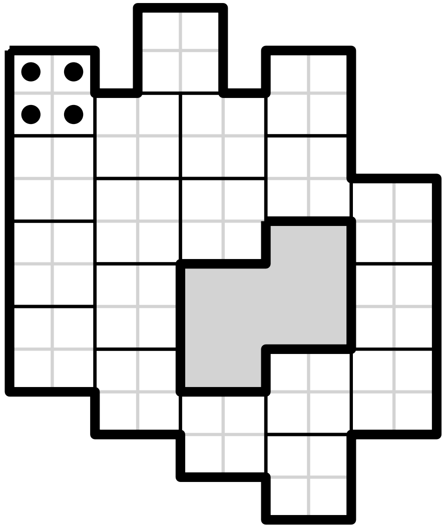

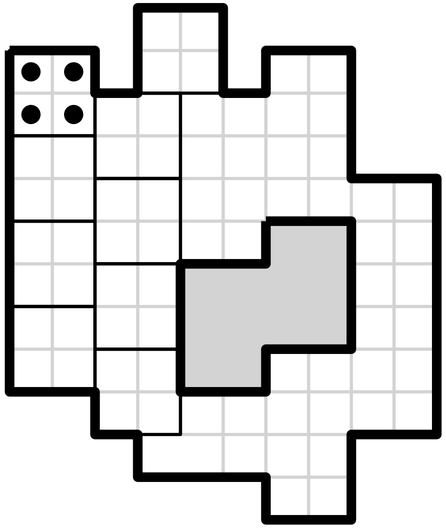

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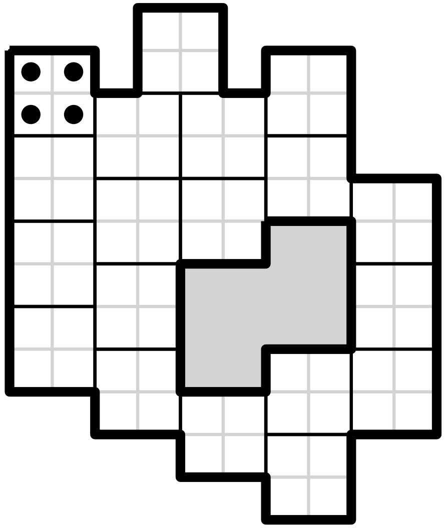

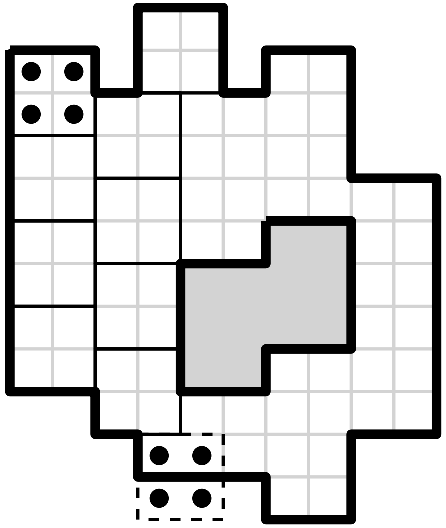

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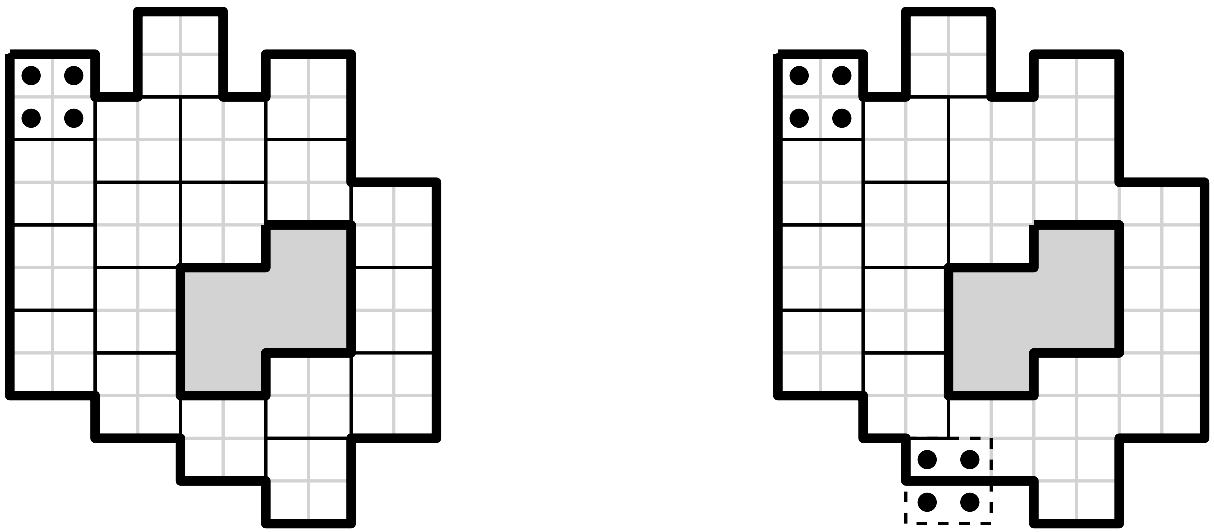
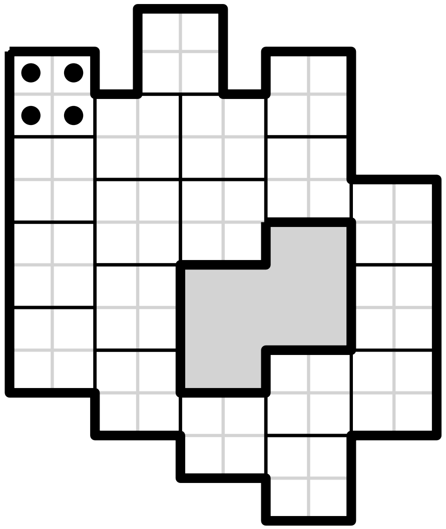
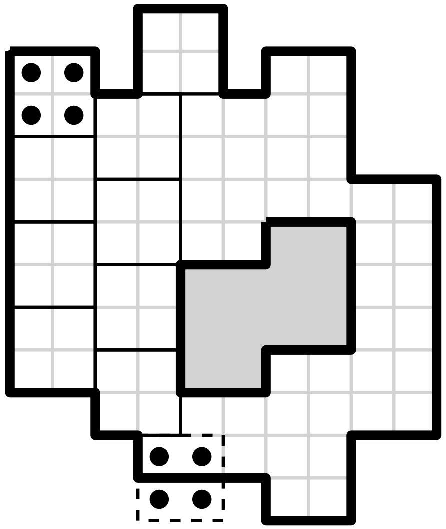


Can be done in $O(A)$ time.

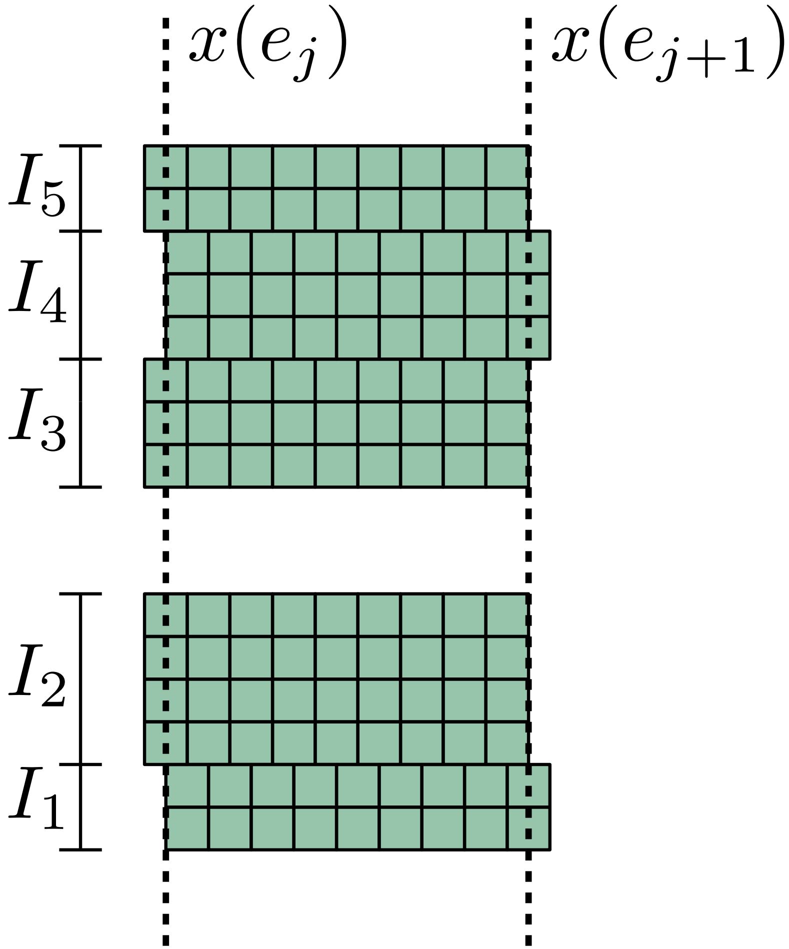
Tiling with 2×2 squares



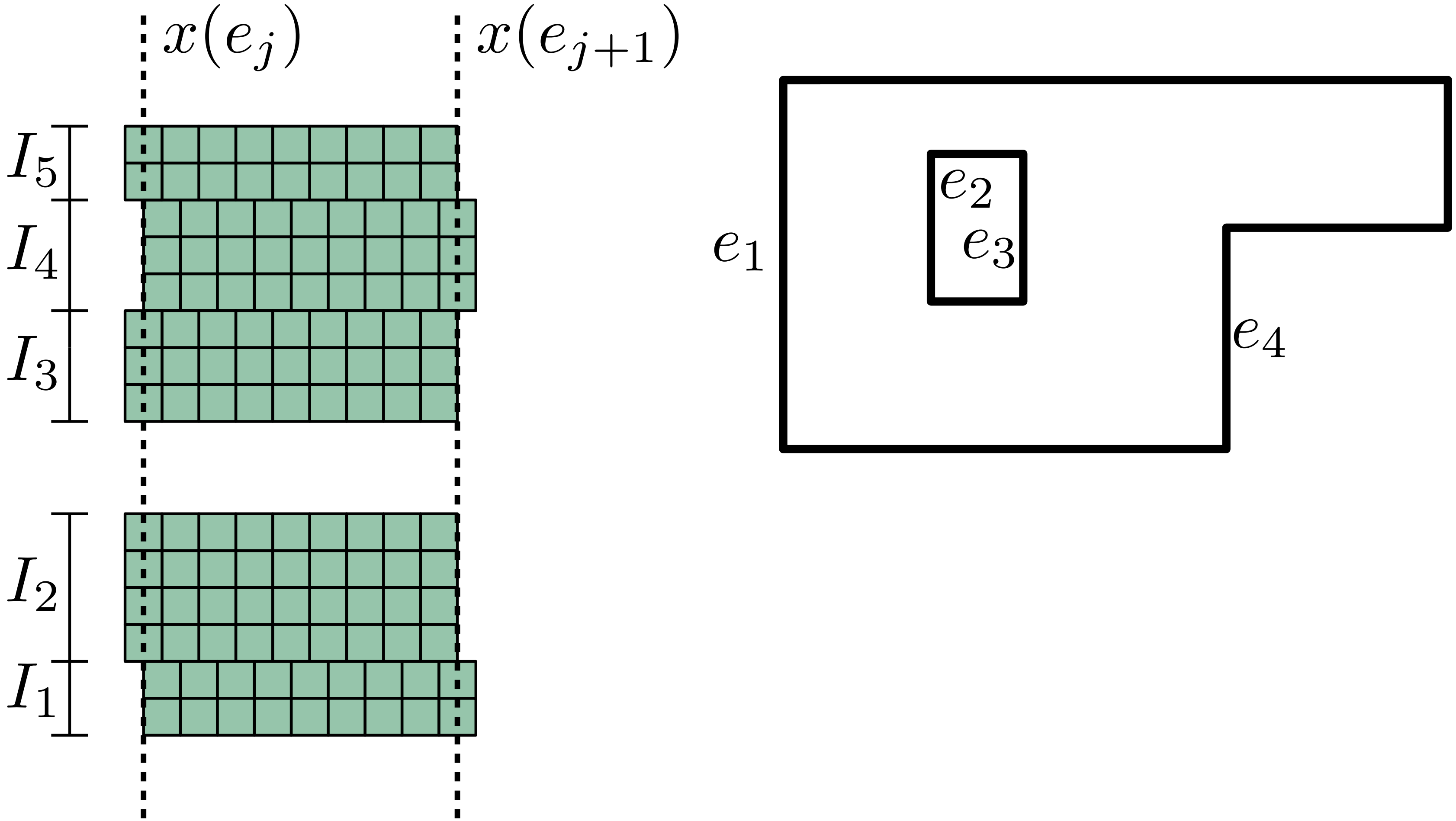


Can be done in $O(A)$ time.

Polynomial-time algorithm but in the area of P!



$x(e_j)$
$x(e_{j+1})$
I_5
I_4
I_3
I_2
I_1



$x(e_j)$
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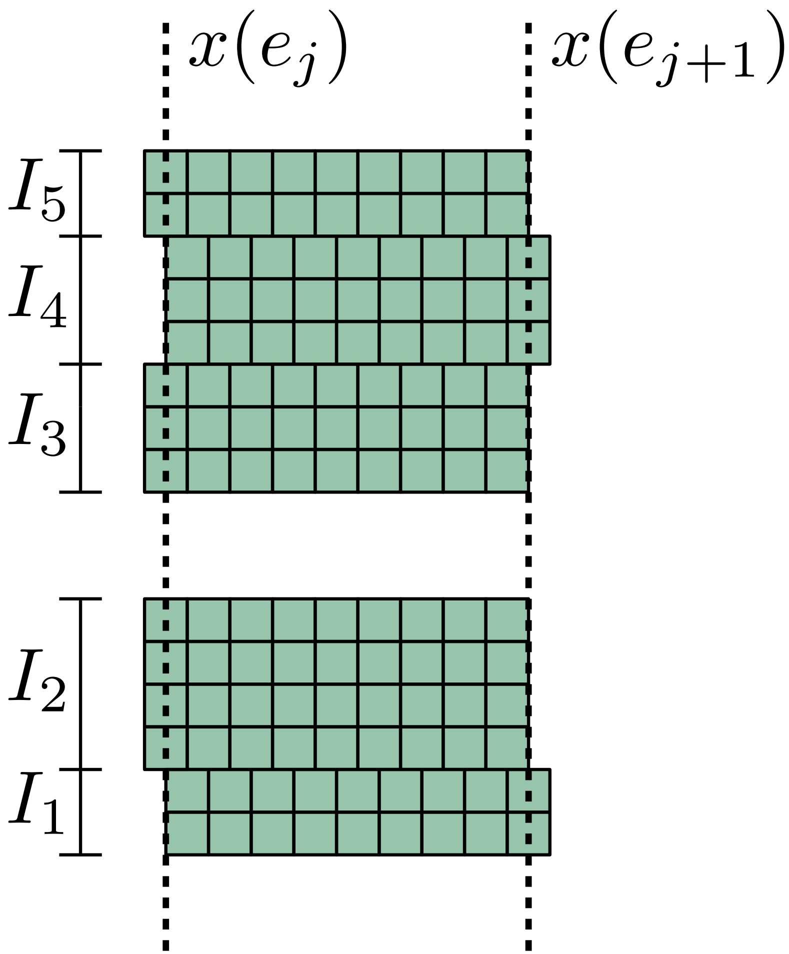
I_4
I_3
I_2
I_1
e_1
e_2
e_3
e_4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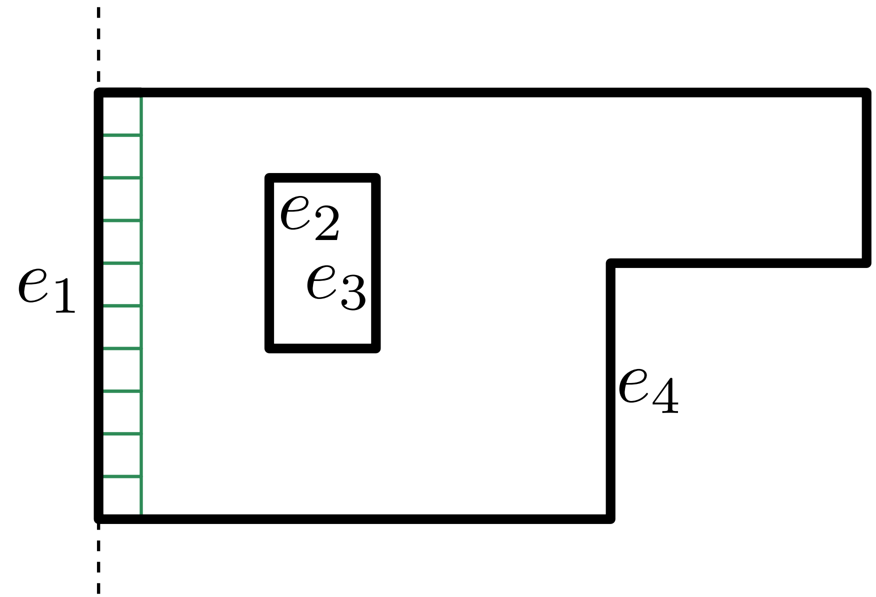
e_1
e_2
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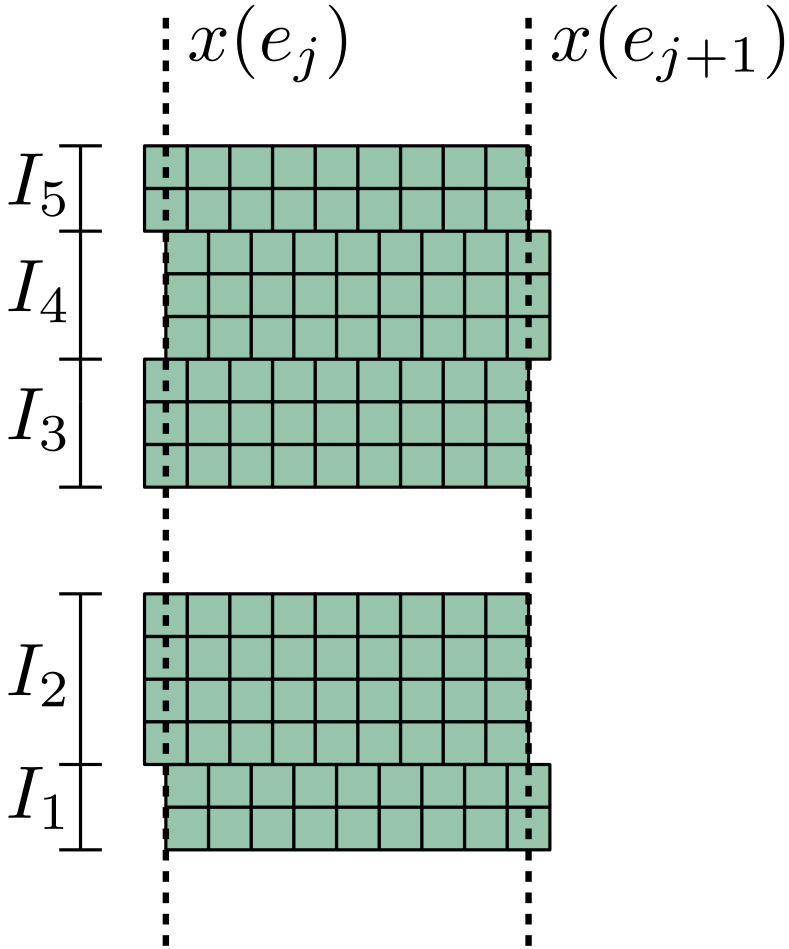

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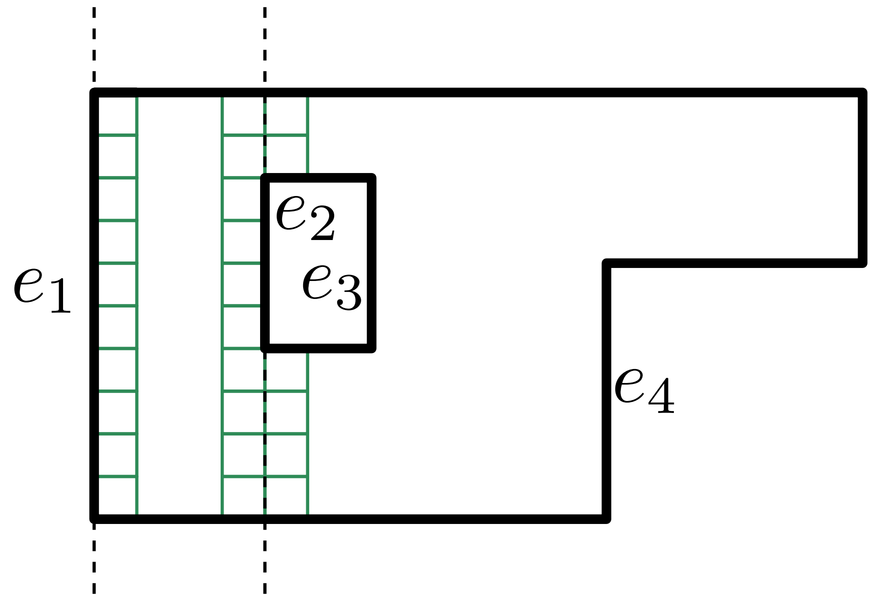

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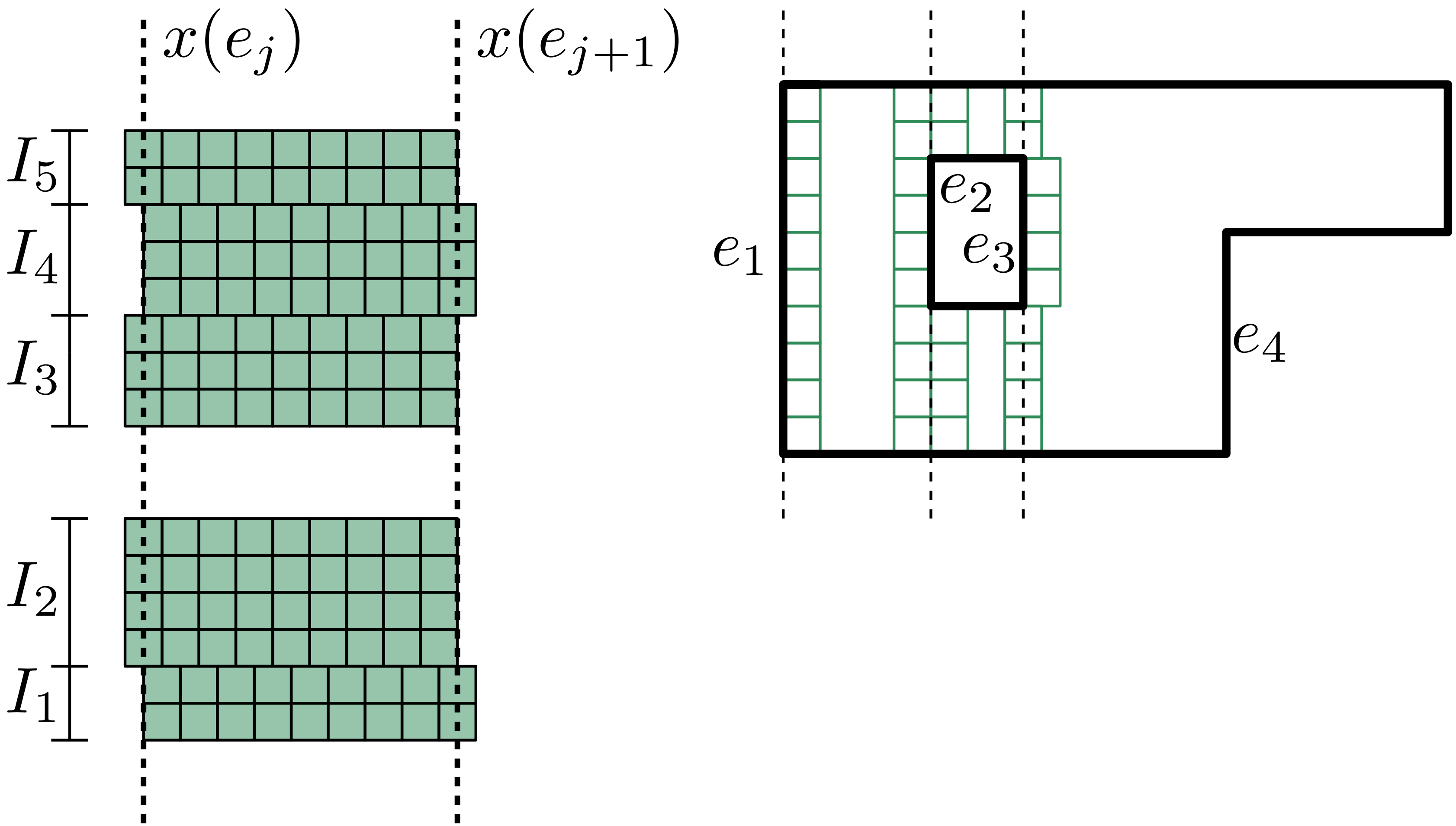

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I_4
I_3
I_2
I_1
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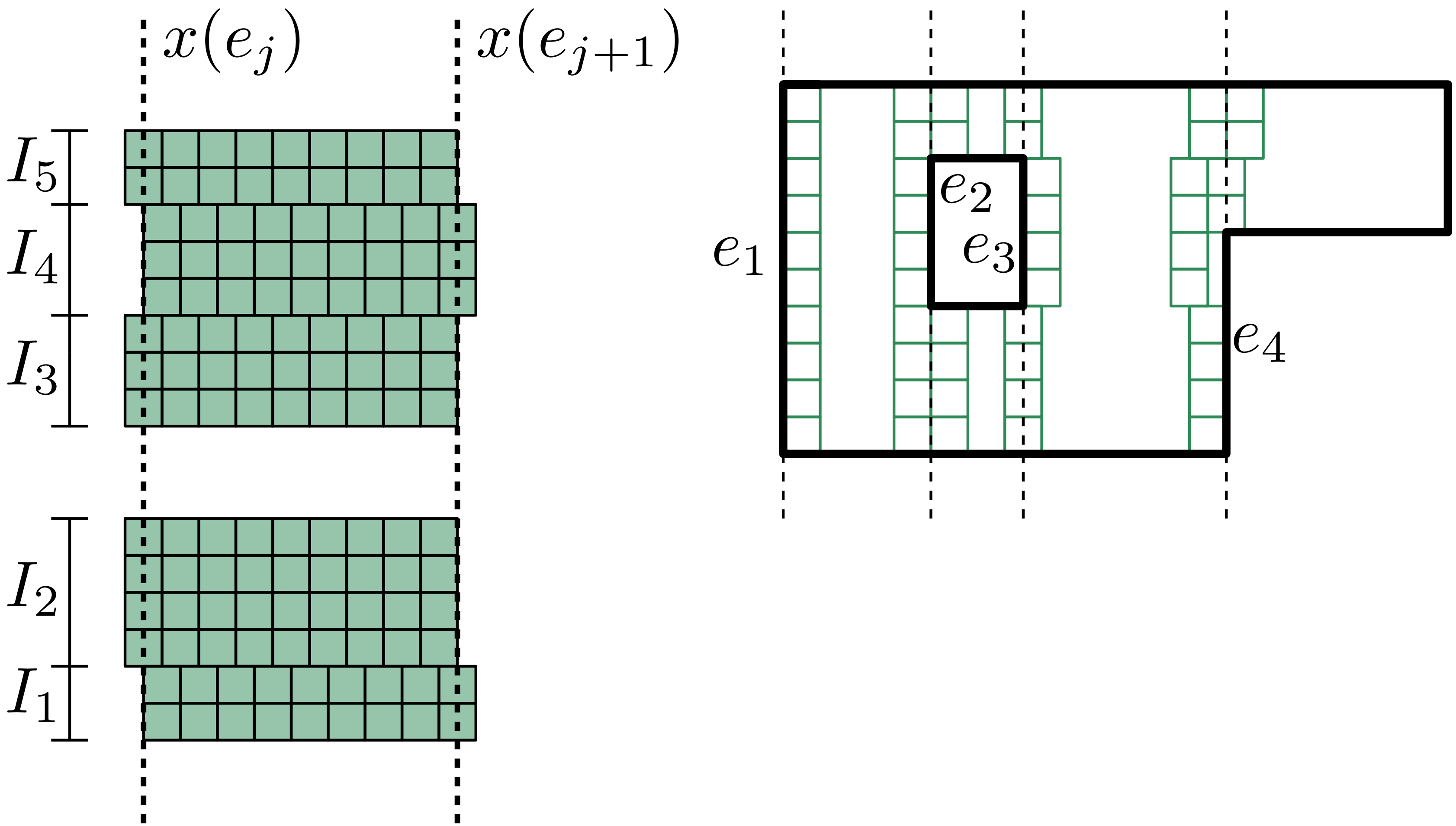
e_3
e_4



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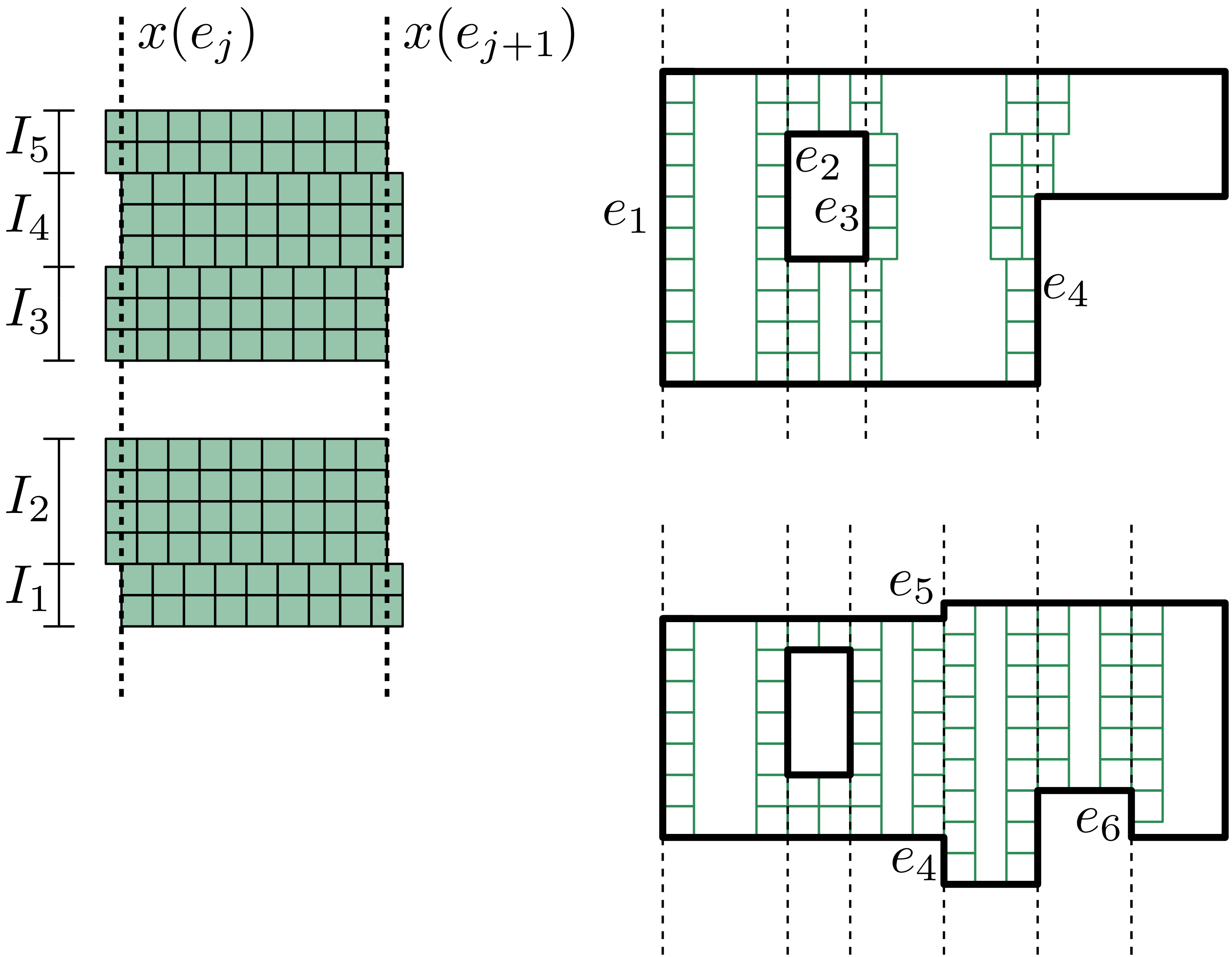
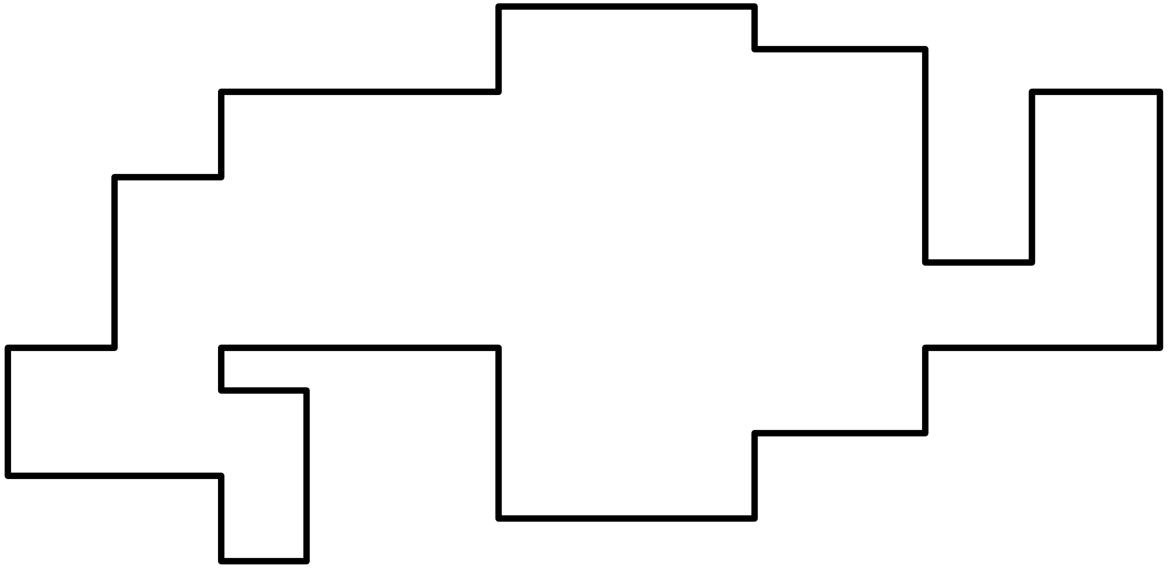
e_1
e_2
e_3
e_4



$x(e_j)$
$x(e_{j+1})$
I_5
I_4
I_3
I_2
I_1
e_1
e_2
e_3
e_4
e_5
e_4
e_6

No holes: $O(n)$ time!



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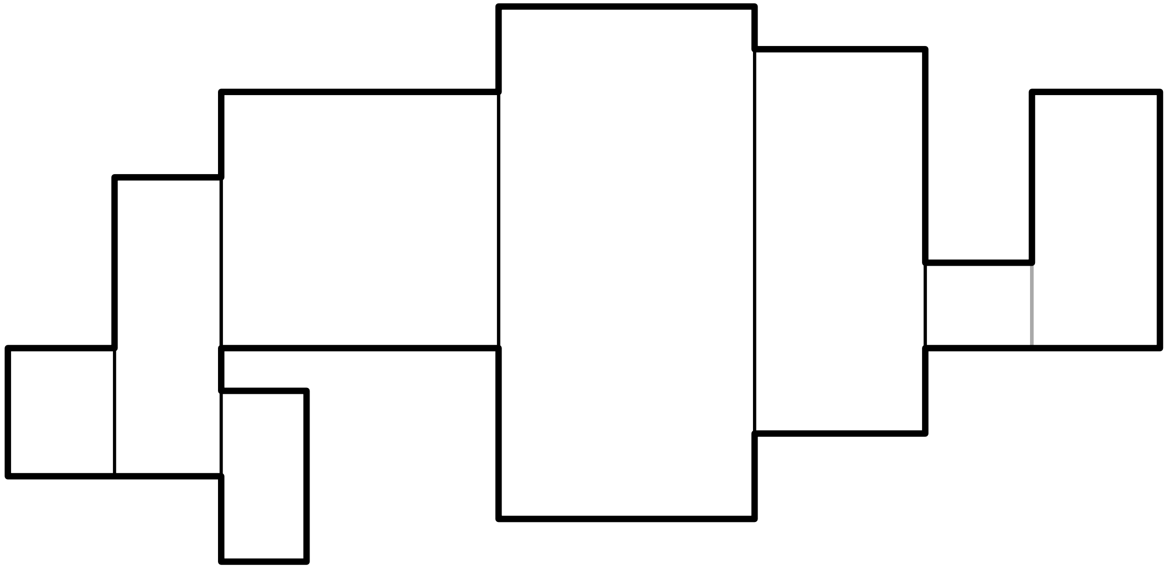

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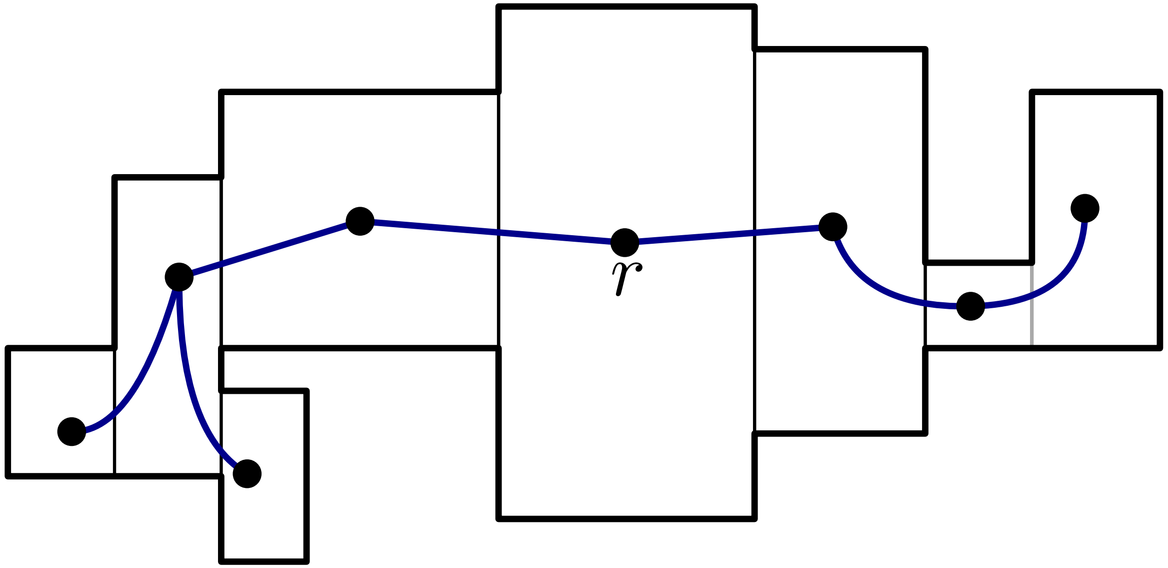

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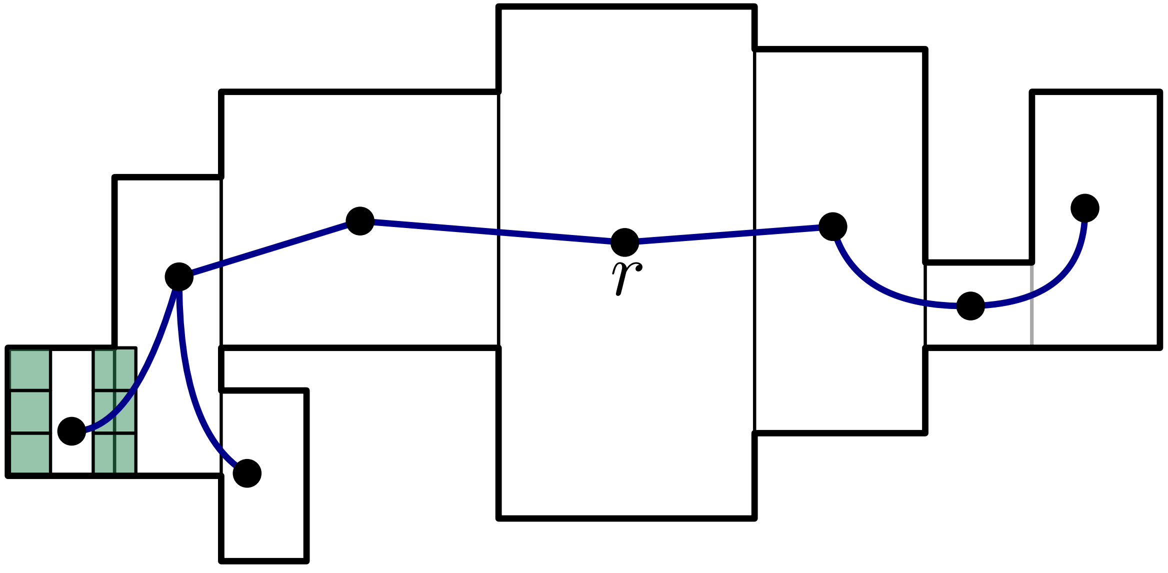

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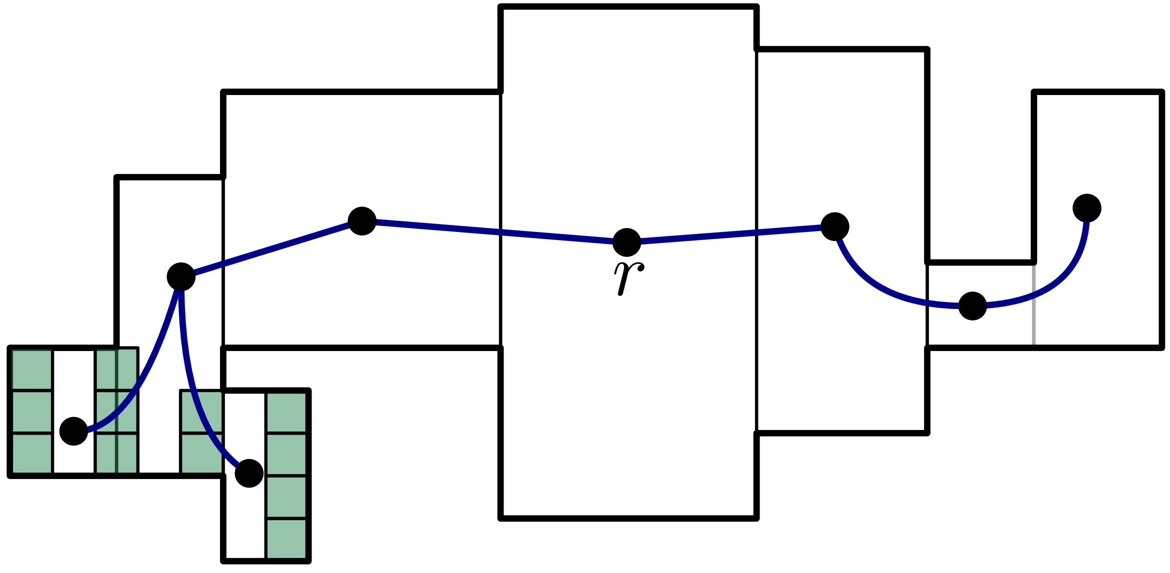

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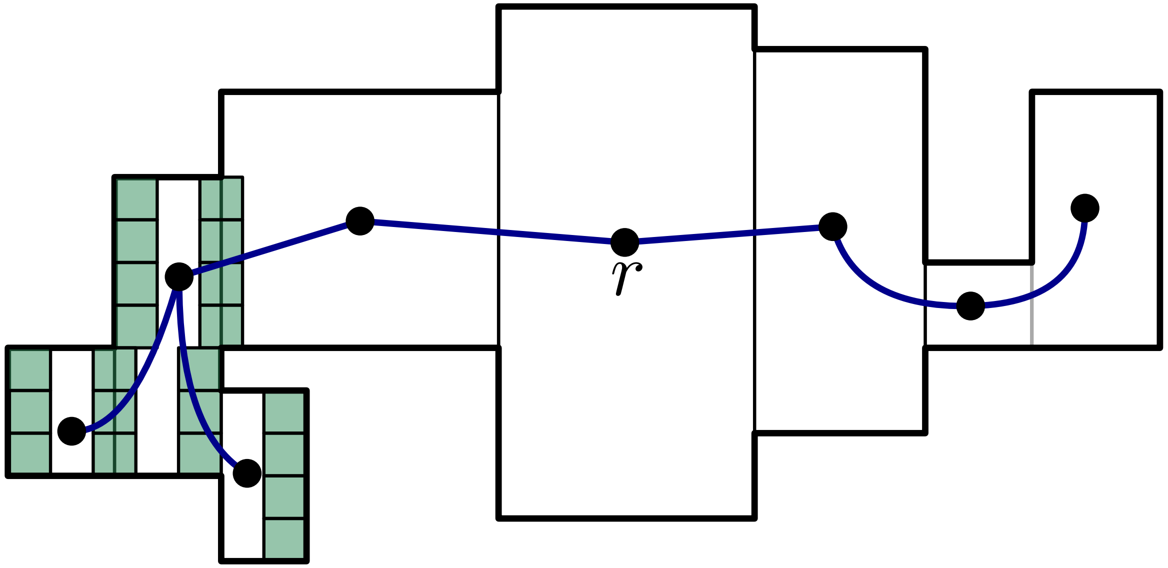

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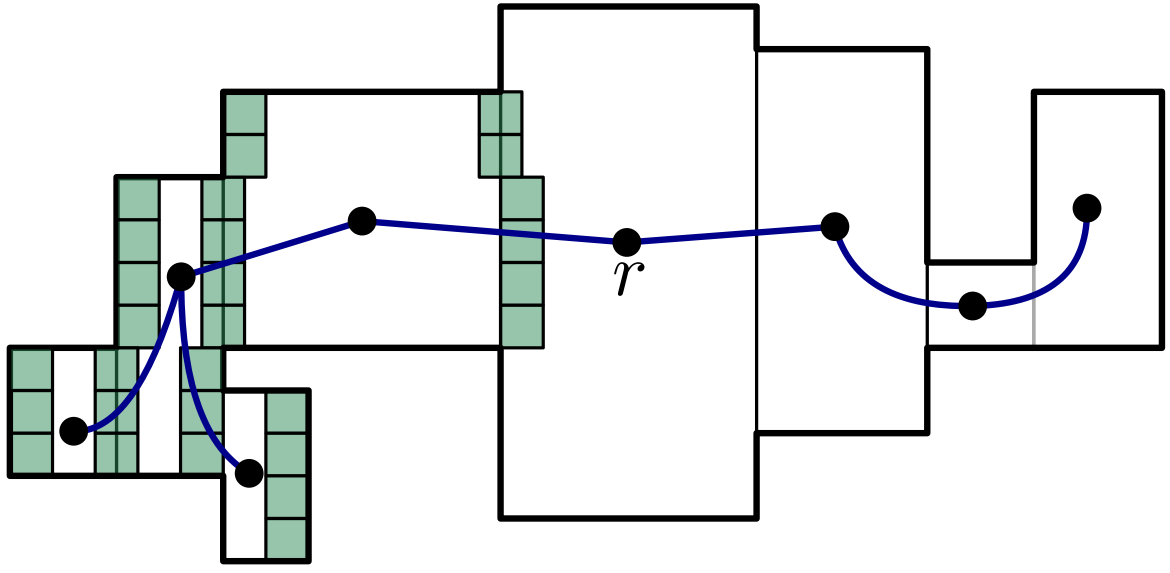

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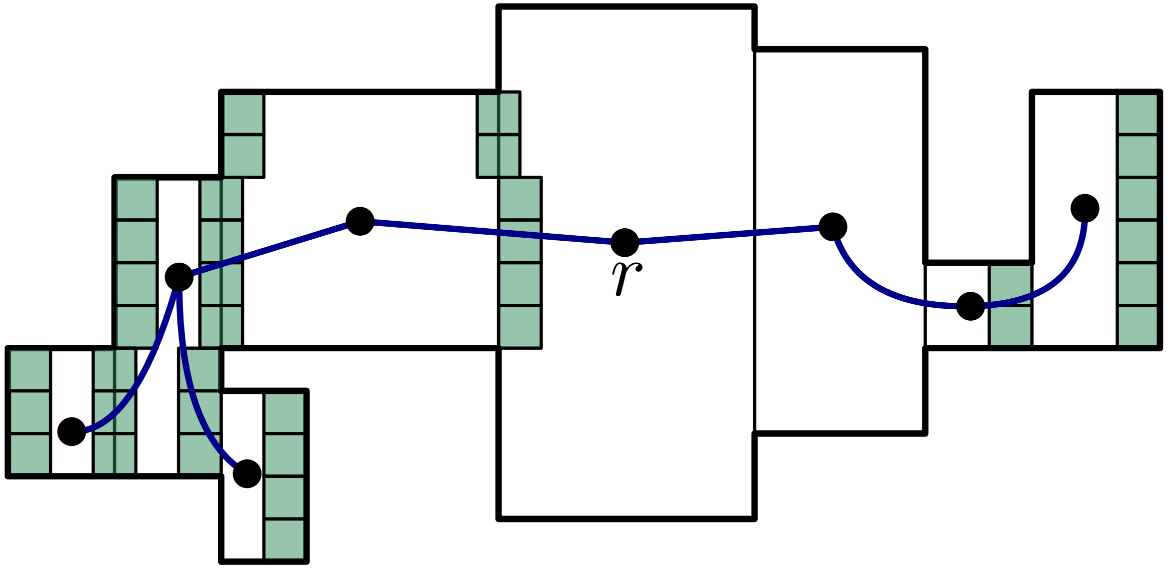

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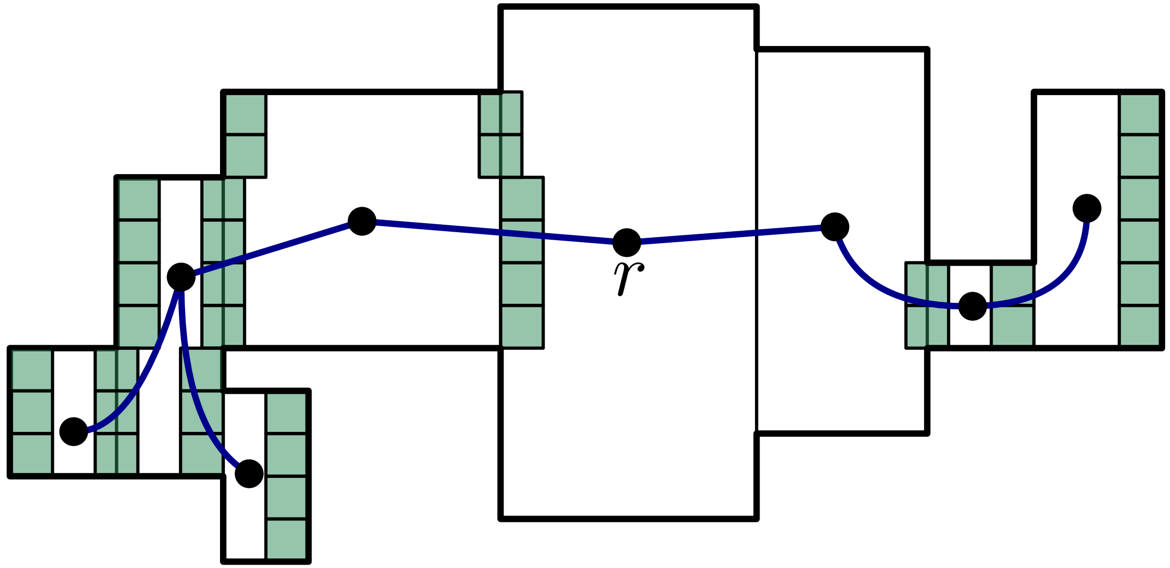

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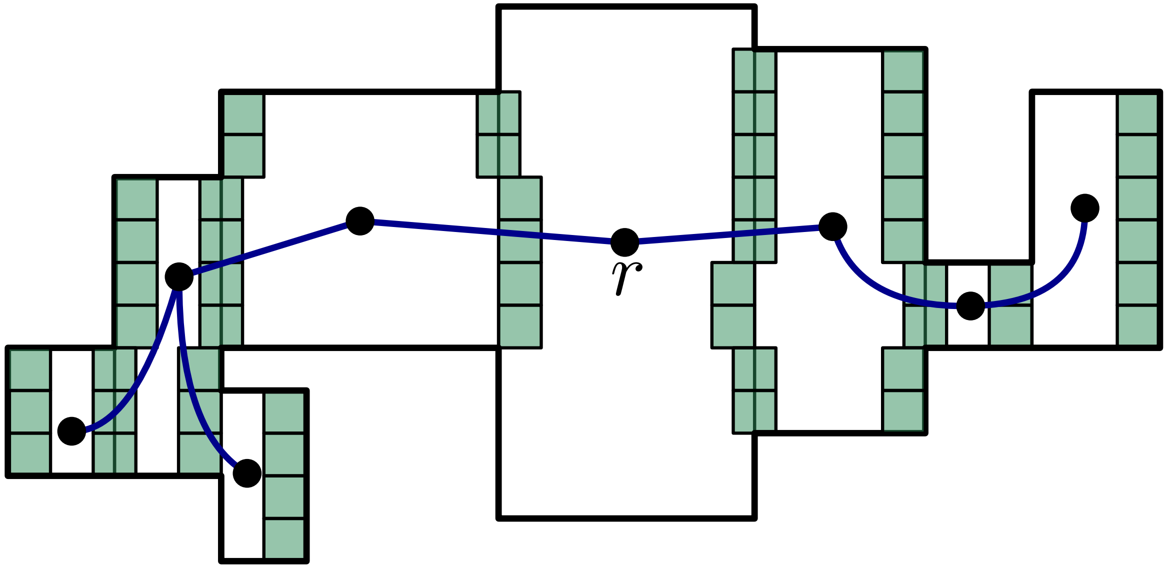

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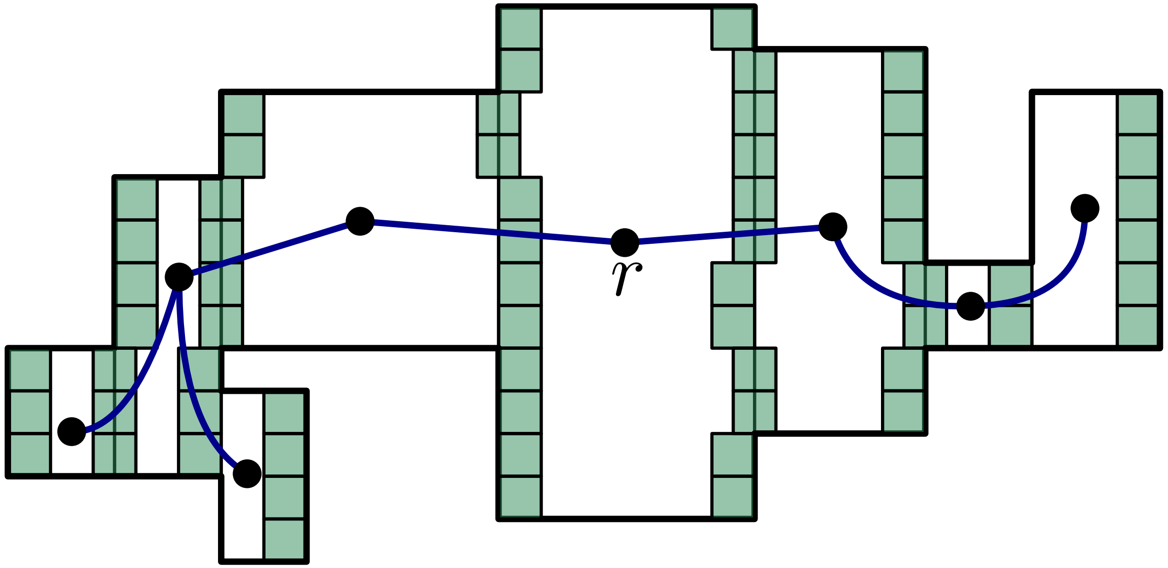

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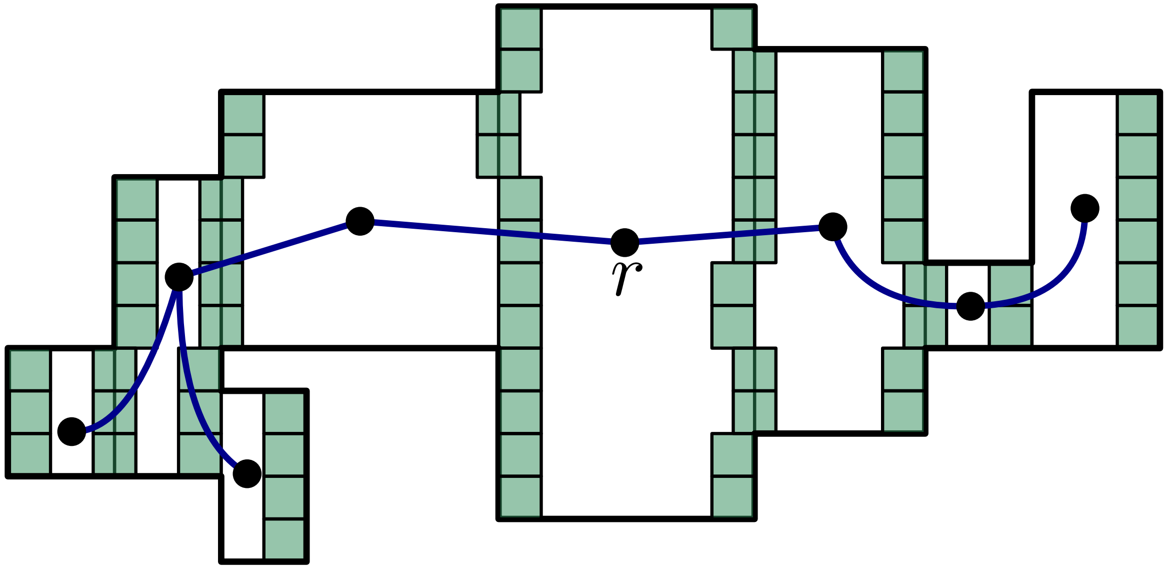
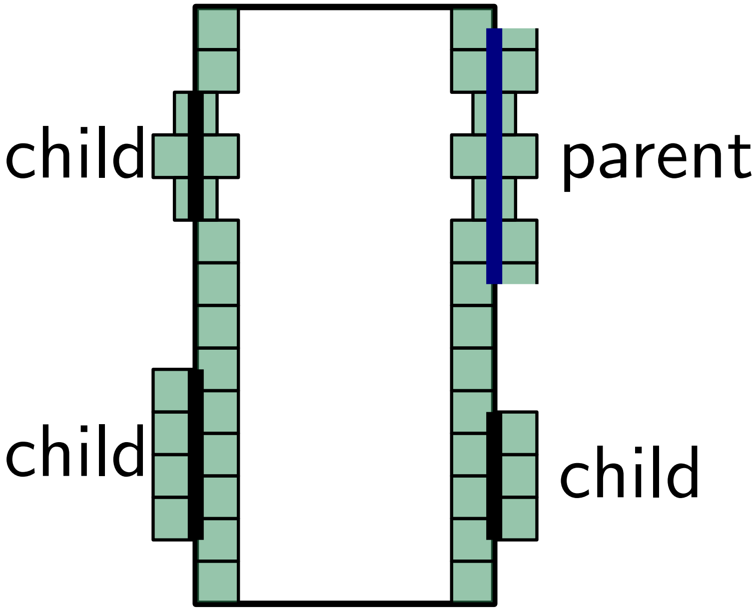
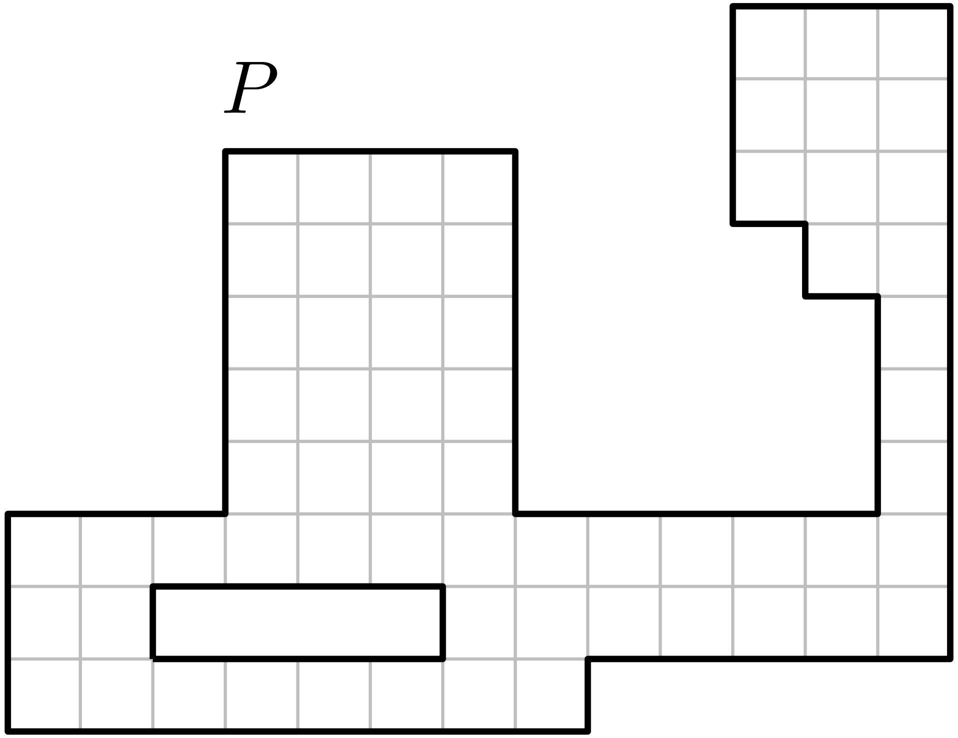


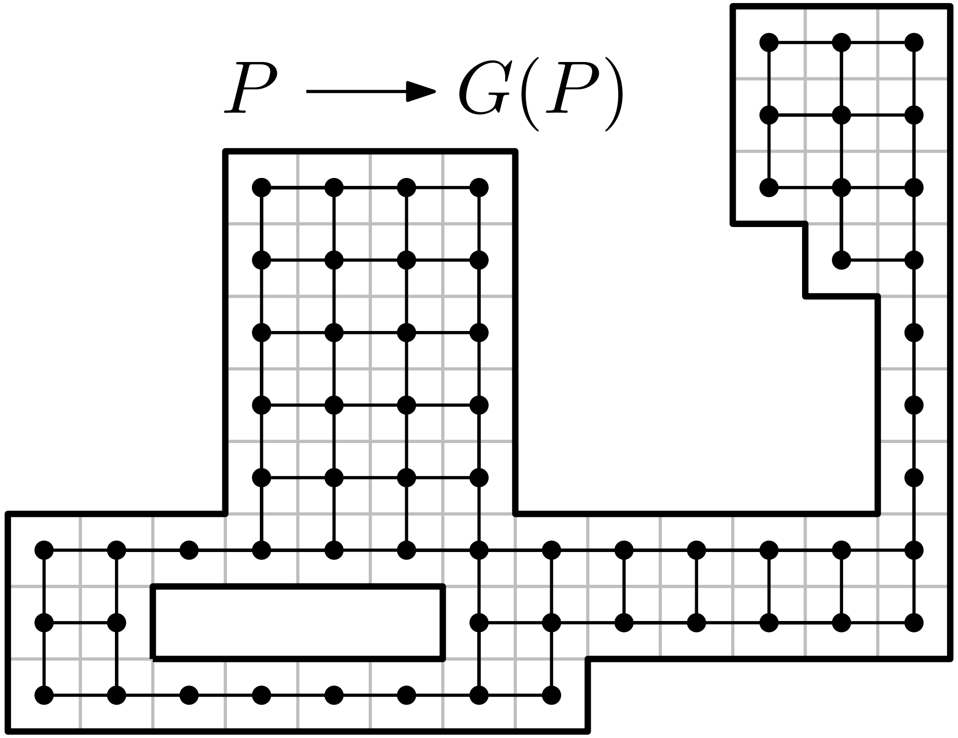


Packing dominos



Packing dominos

$P \longrightarrow G(P)$



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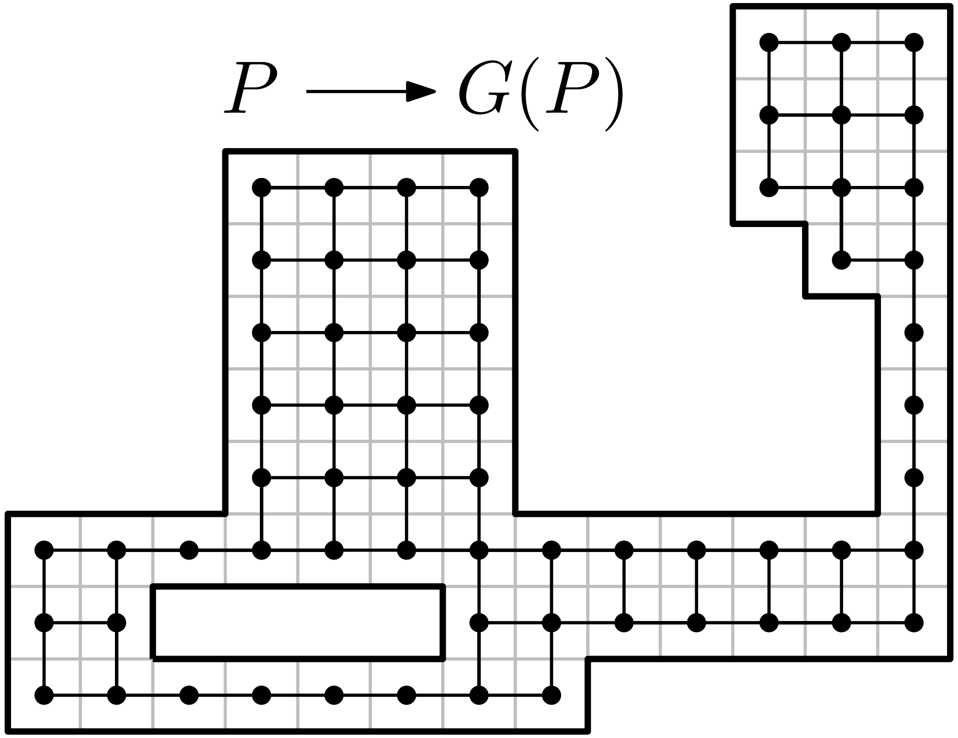
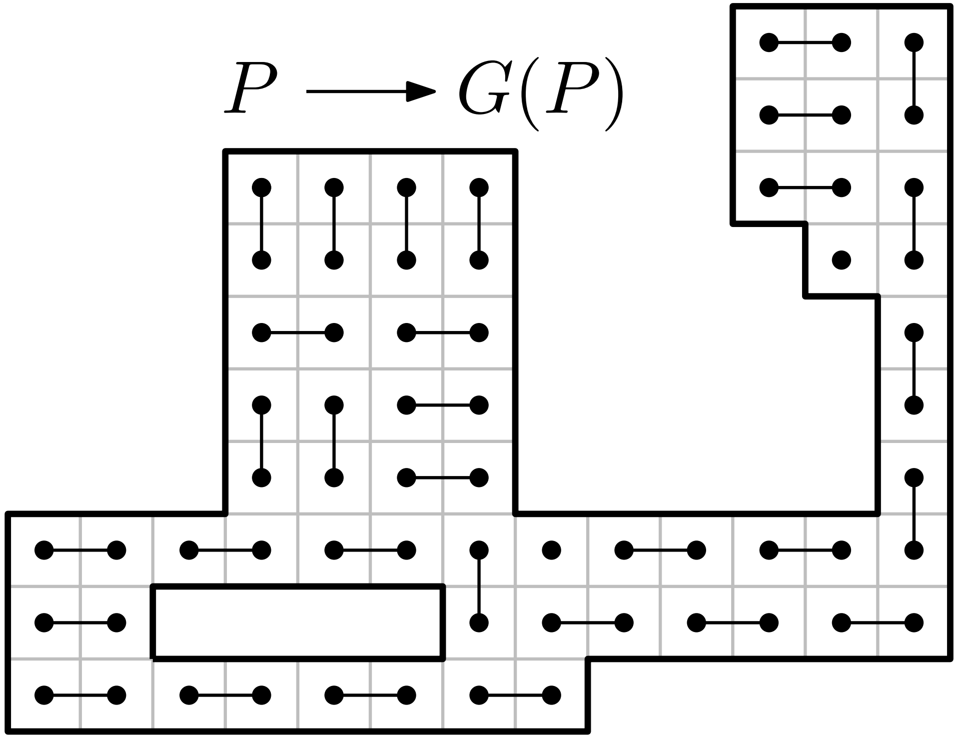


Maximum domino packing of $P \leftrightarrow$ Maximum matching of $G(P)$

Packing dominos



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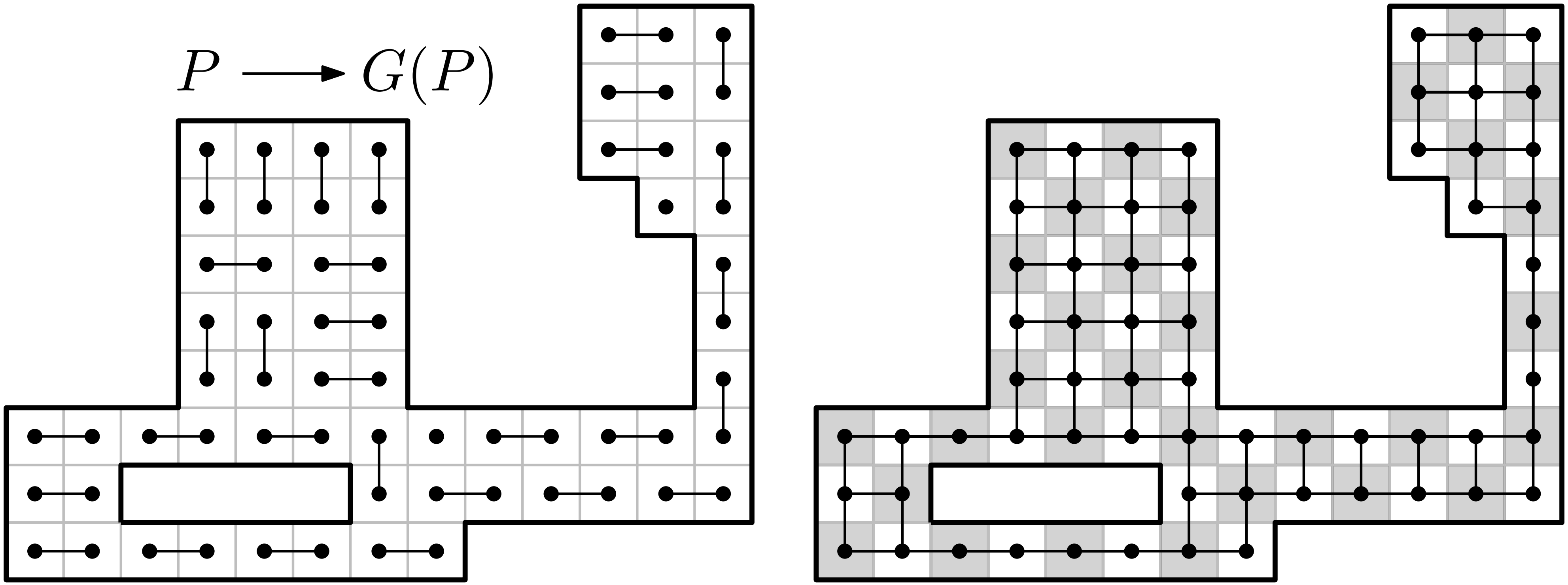
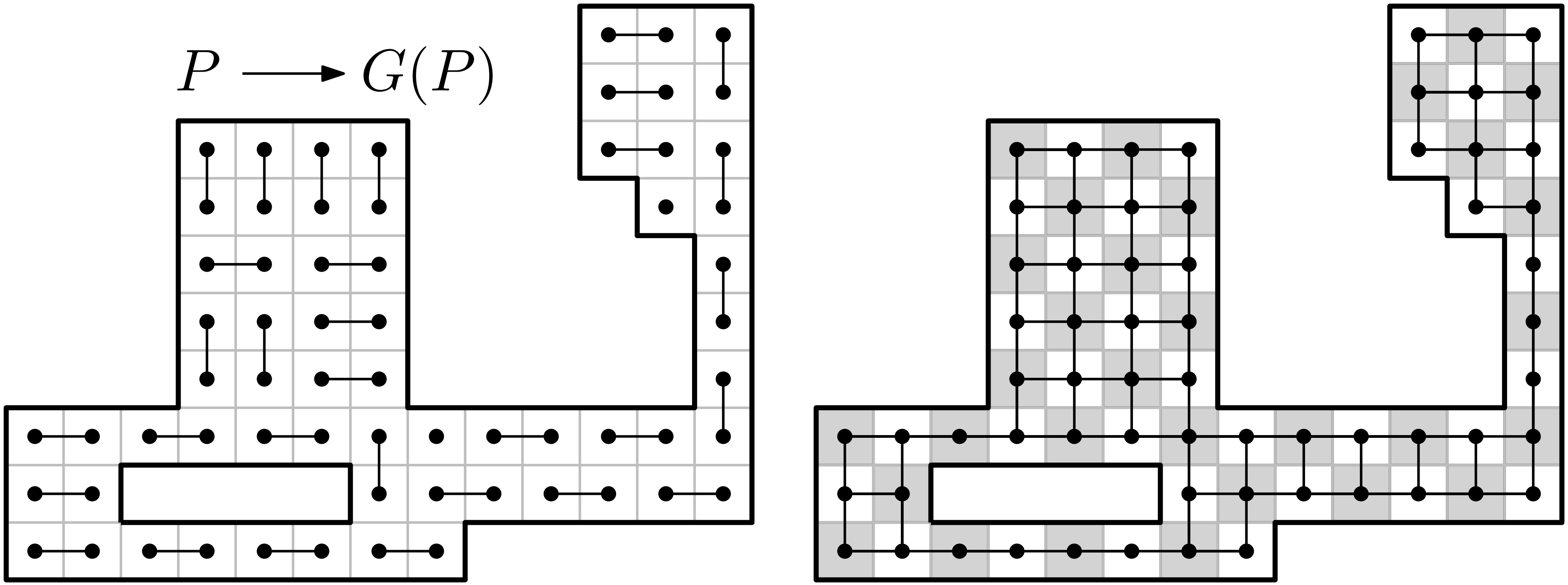
Packing dominos



Maximum domino packing of $P \leftrightarrow$ Maximum matching of $G(P)$

Time $O(A^{3/2})$ for maximum domino packing using Hopcroft-Karp, where A is the *area* of P (Berman et al. '82)

Packing dominos



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Multiple source multiple sink maximum flow: $\widetilde{O}(A)$ [Borradaile et al., SICOMP 2017].

Related work

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Combinatorial Group Theory approach for deciding tileability.

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Berger '66:
Deciding if a finite set of polyominos can tile the plane is Turing-complete

This Talk

Packing Dominos in $\widetilde{O}(n^3)$ time

This Talk

Packing Dominos in $\widetilde{O}(n^3)$ time

Assume no holes

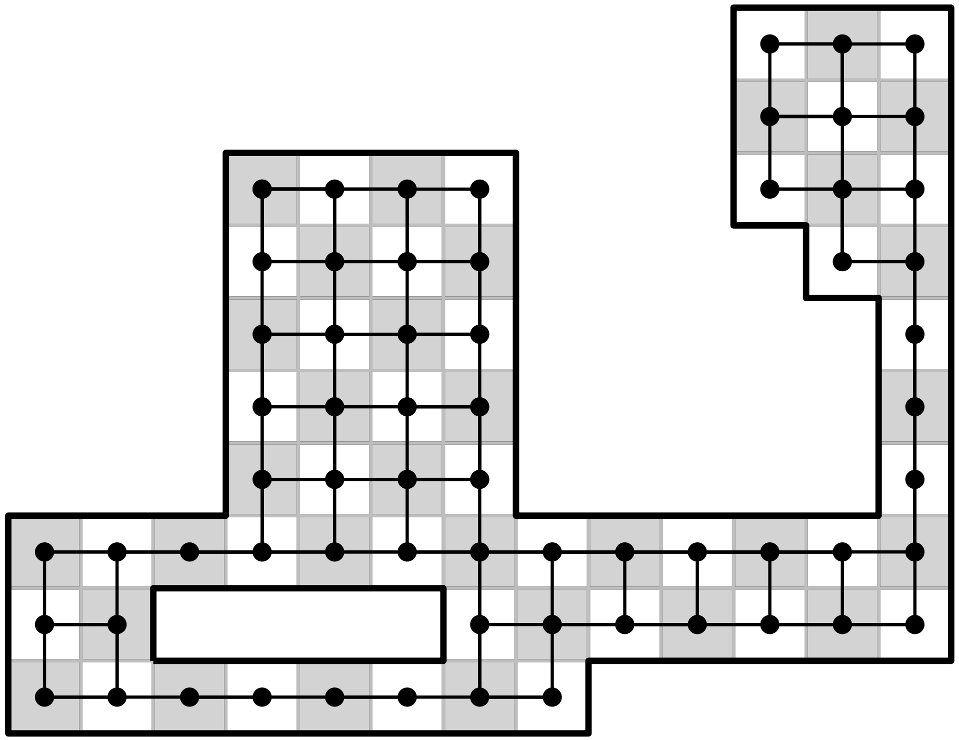
General idea

Ignore parts of P where an optimal packing is trivial and leaves no uncovered squares.

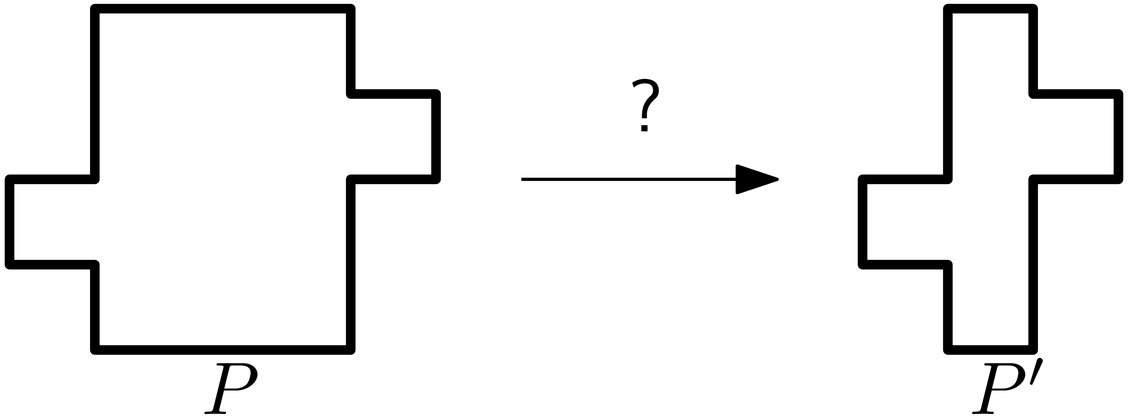
Create graph G^* of size $O(\text{poly } n)$ for the remaining part.

Find maximum matching M in G^*.

Return $|M| + \frac{\text{area}(P) - V(G^*)}{2}$.



Simple algorithm



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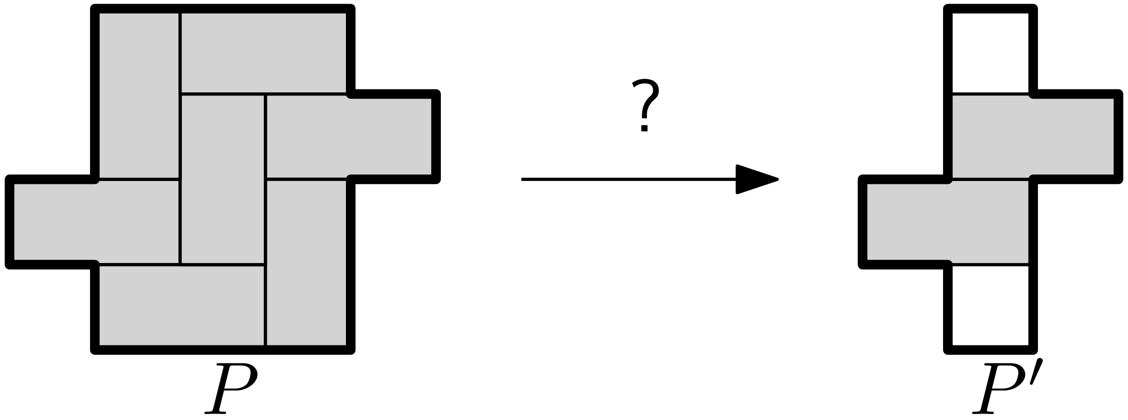

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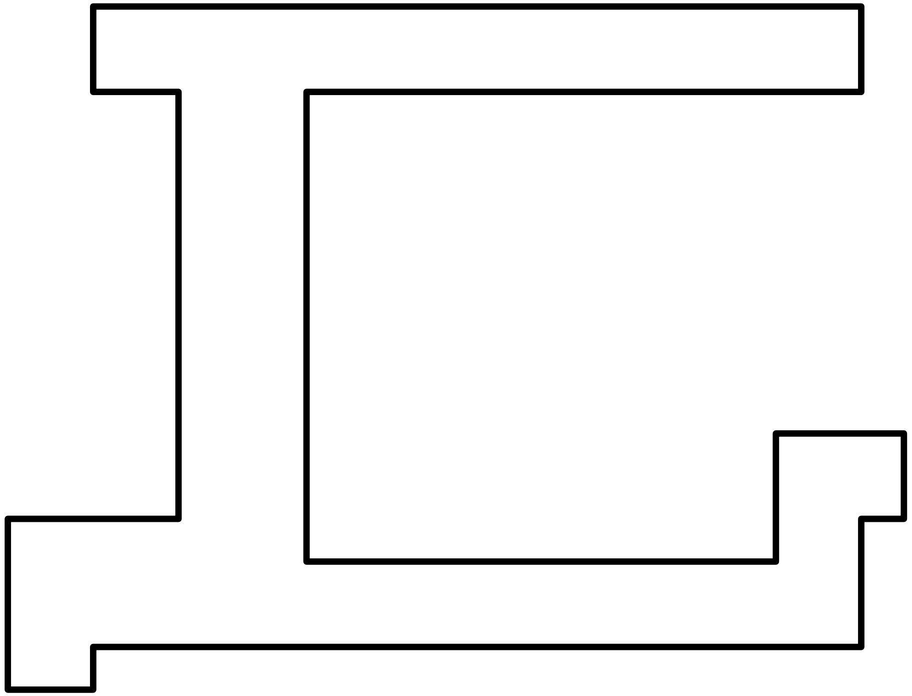

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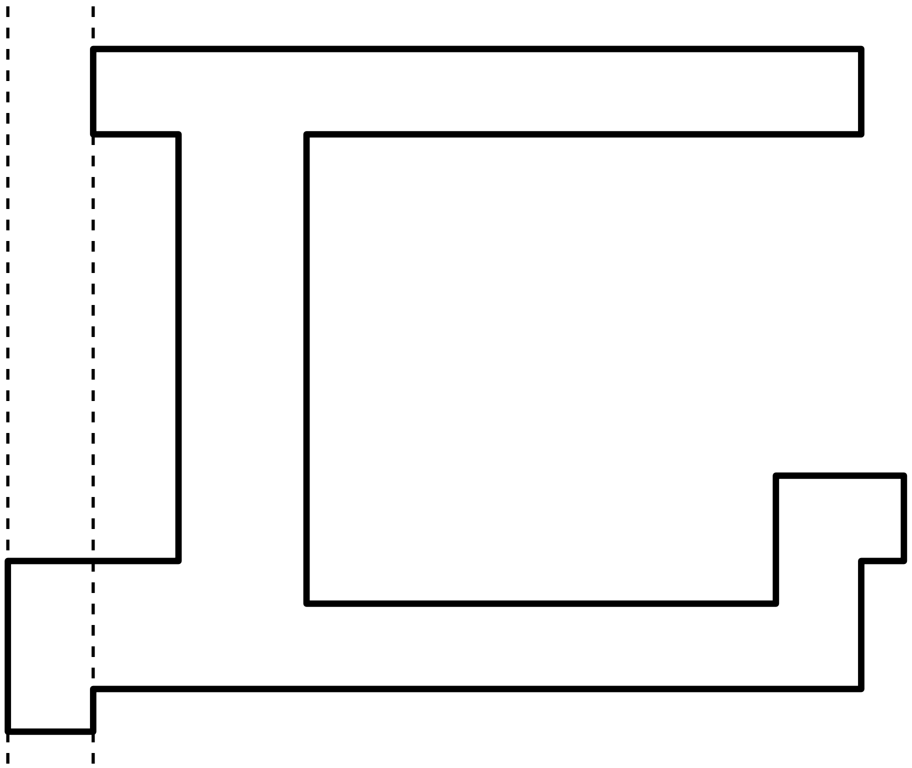

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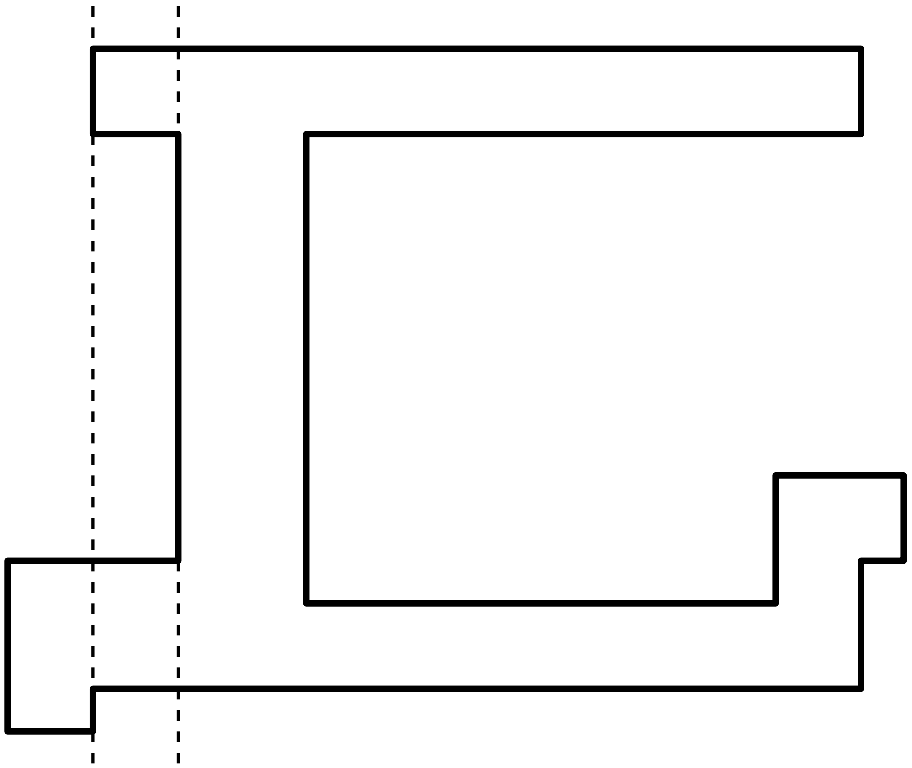

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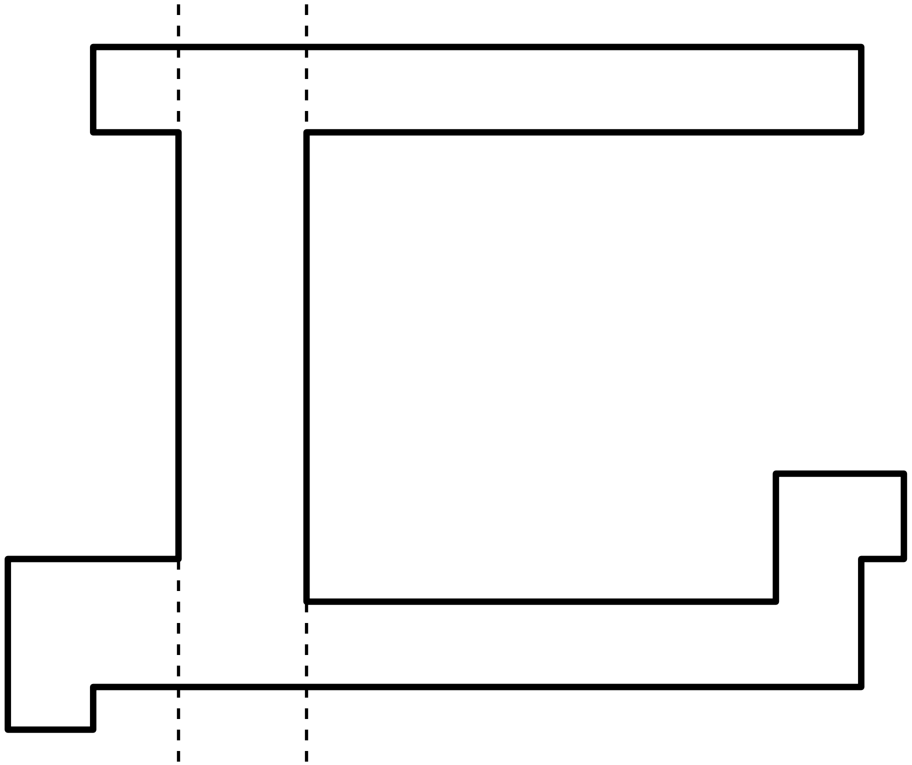

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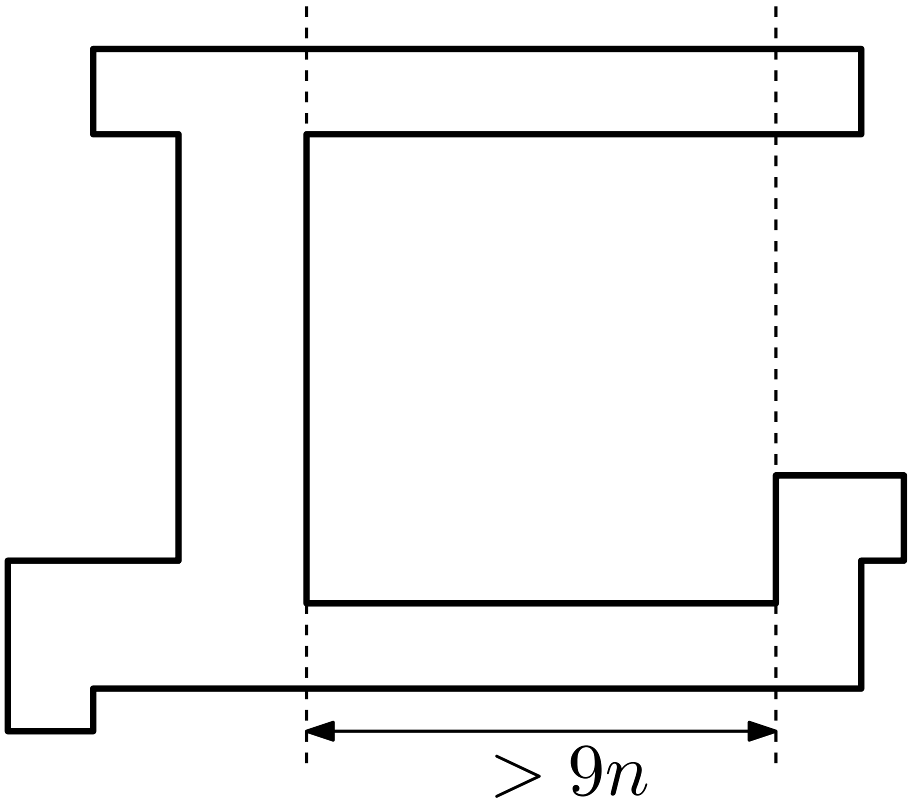

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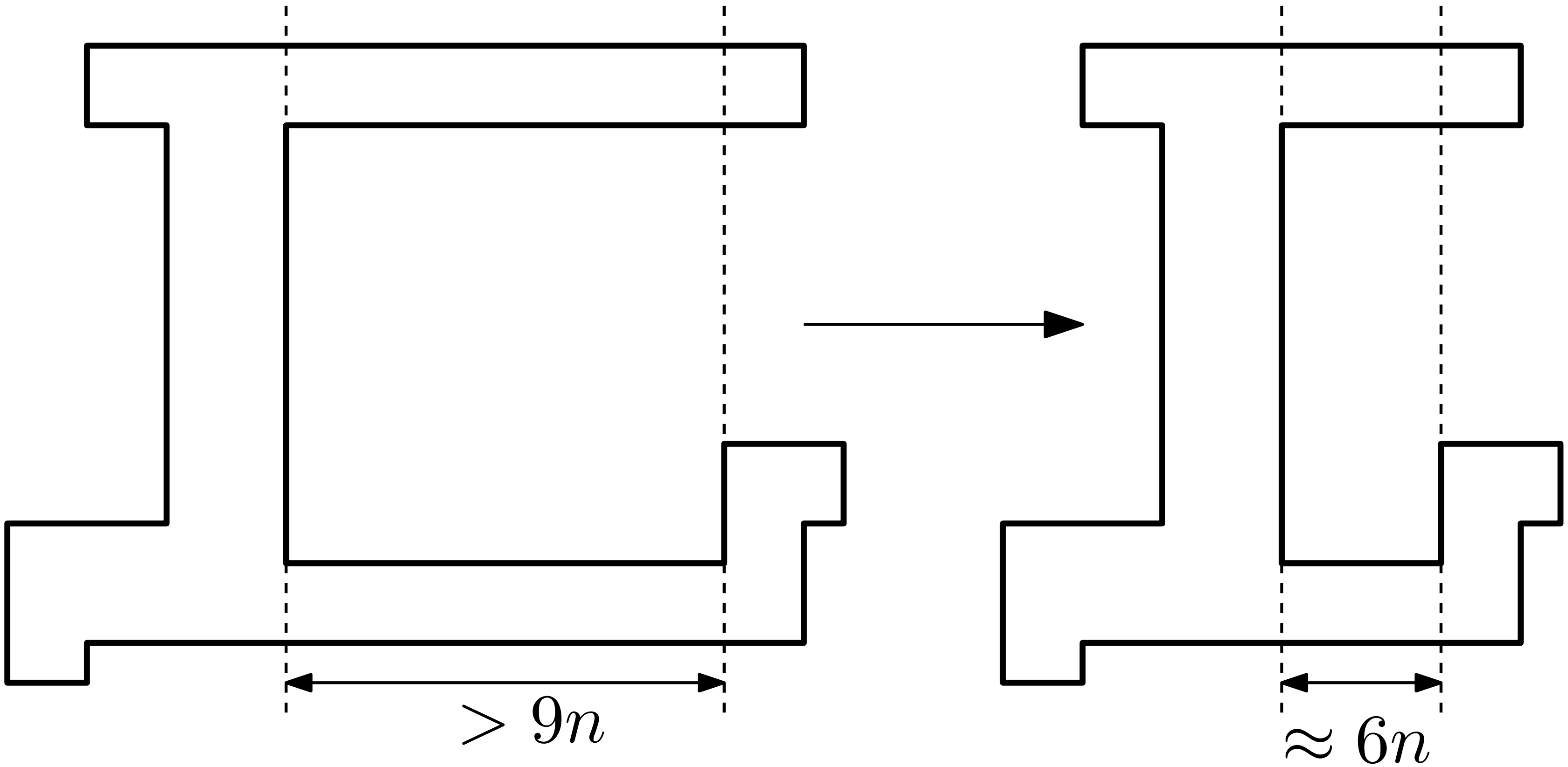

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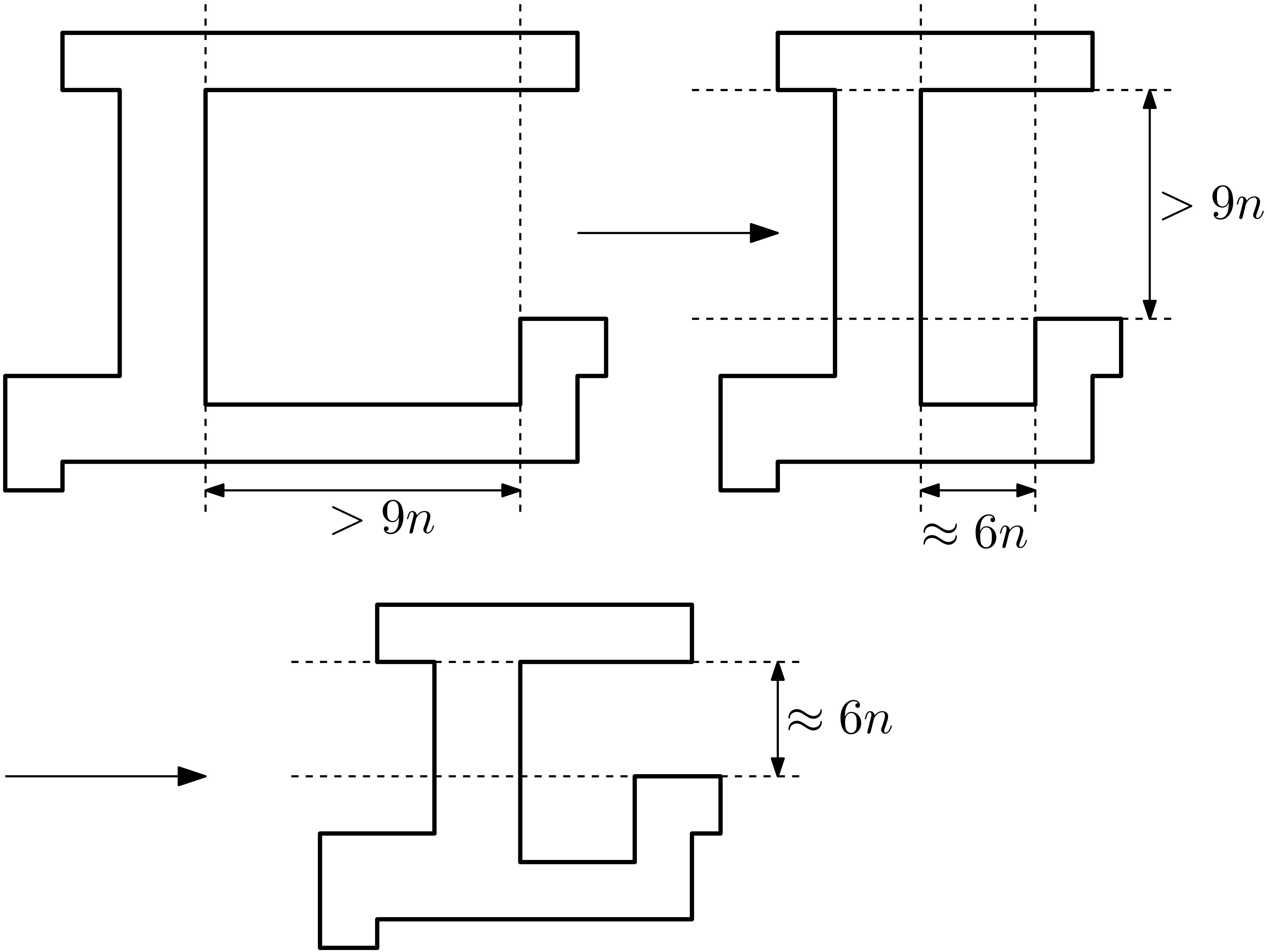

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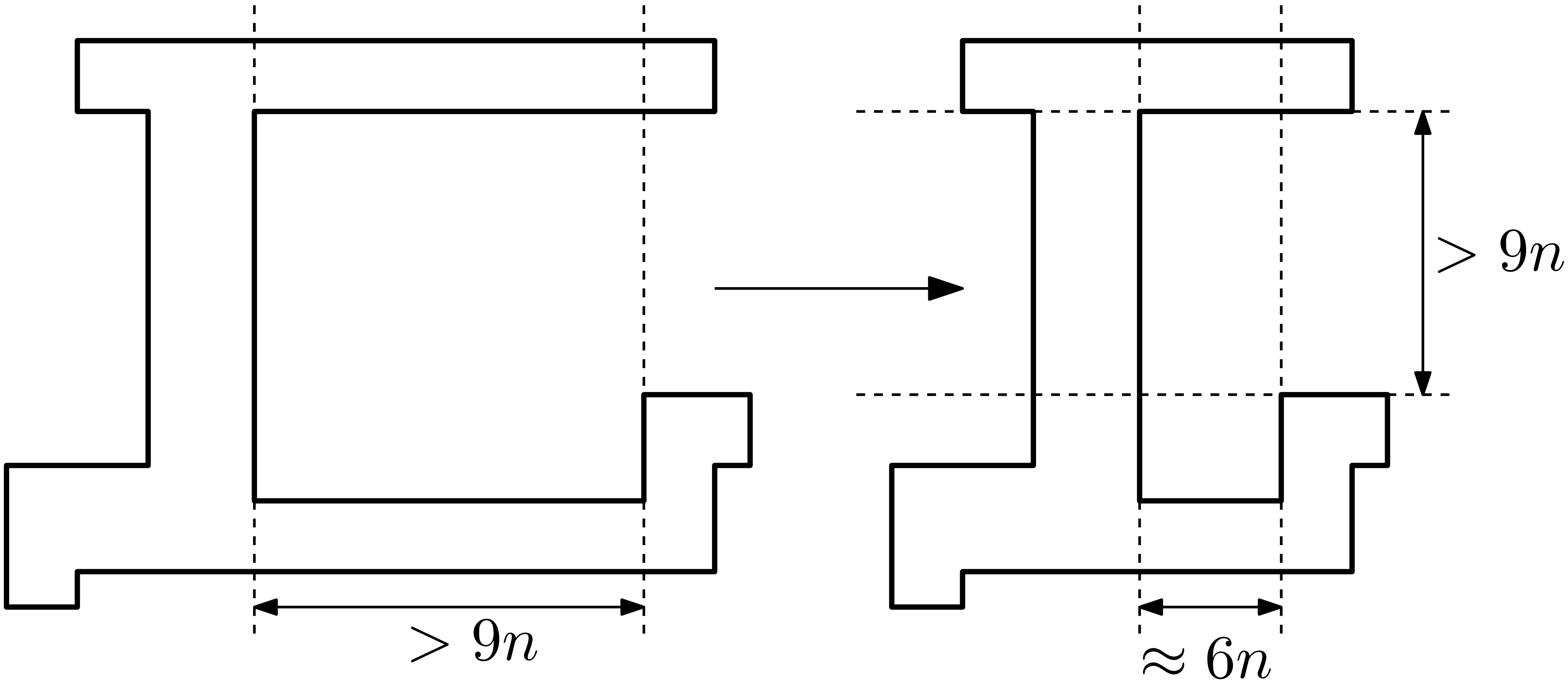
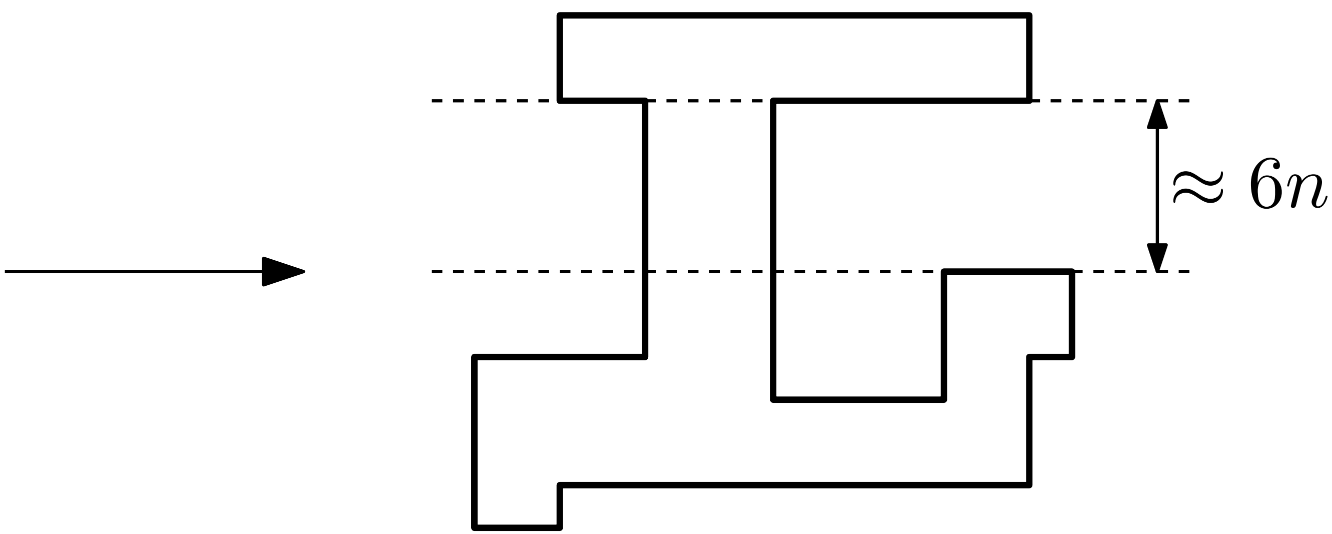
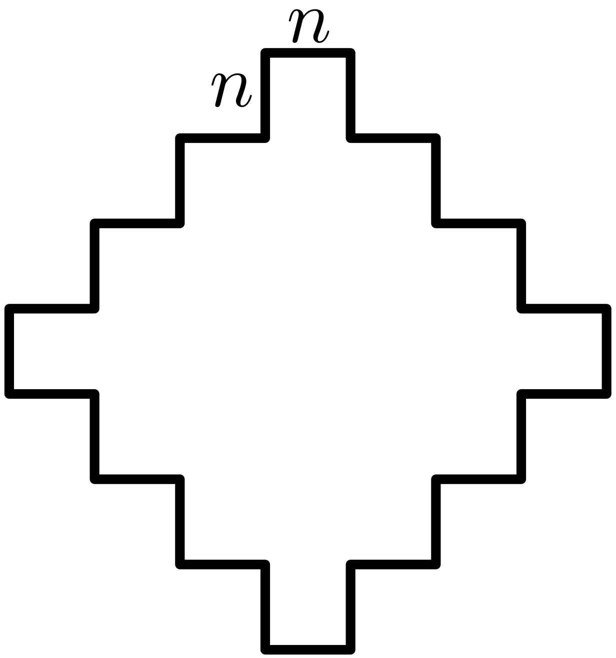




area $O(n^4) \Rightarrow$

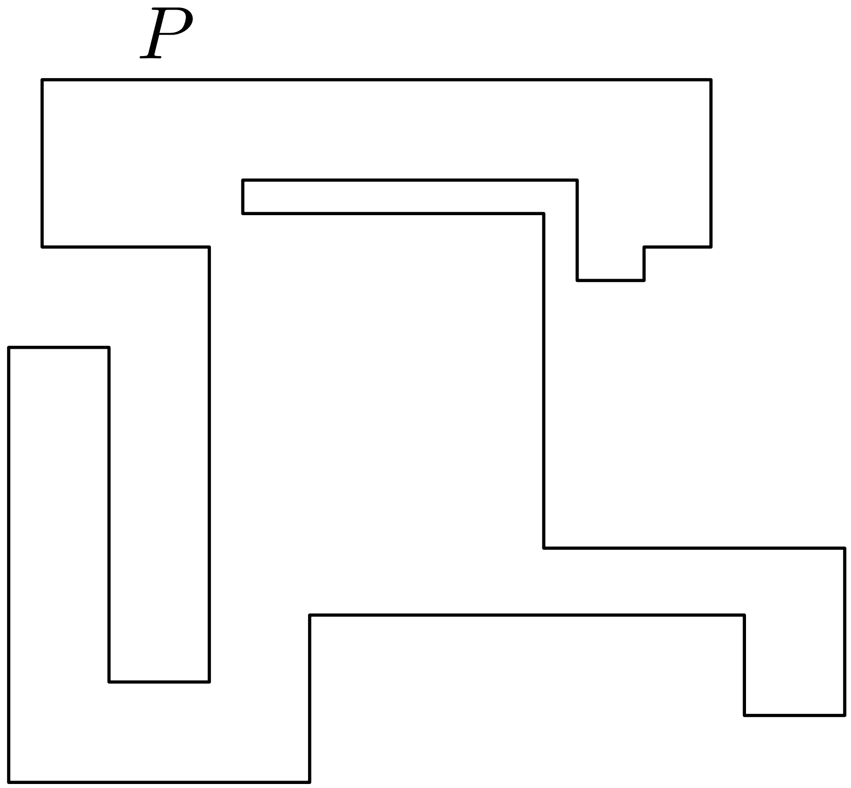
time $\widetilde{O}(n^4)$

Running time of simple algorithm



$$\widetilde{O}(n^4)$$

Cubic-time algorithm (assume no holes)



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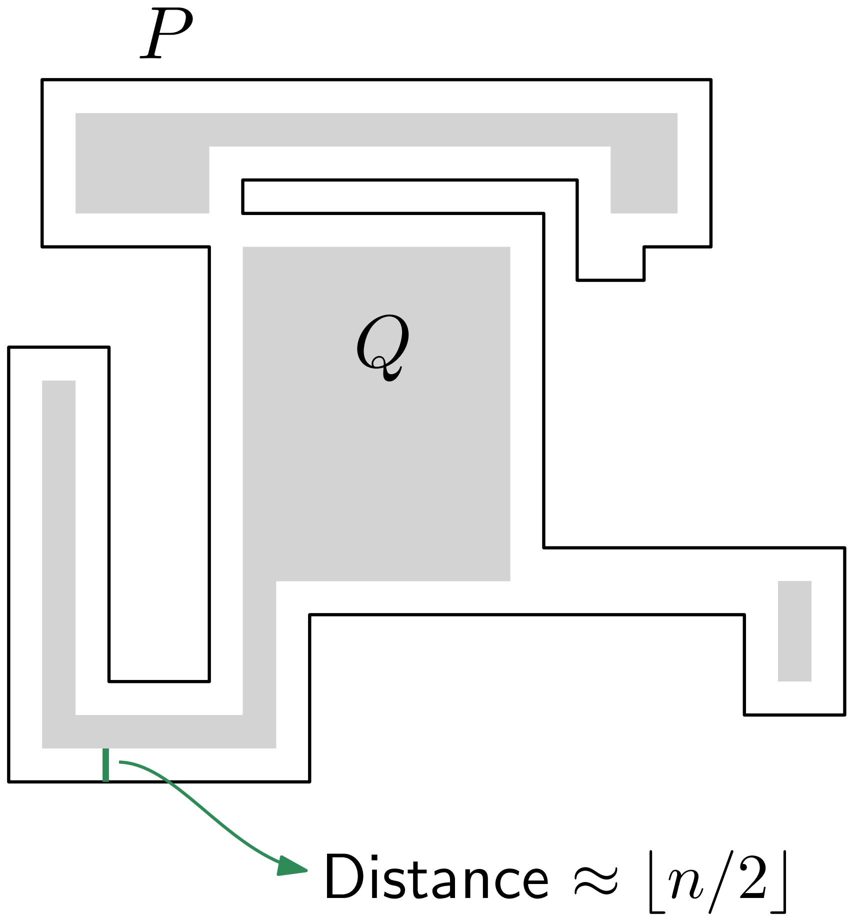

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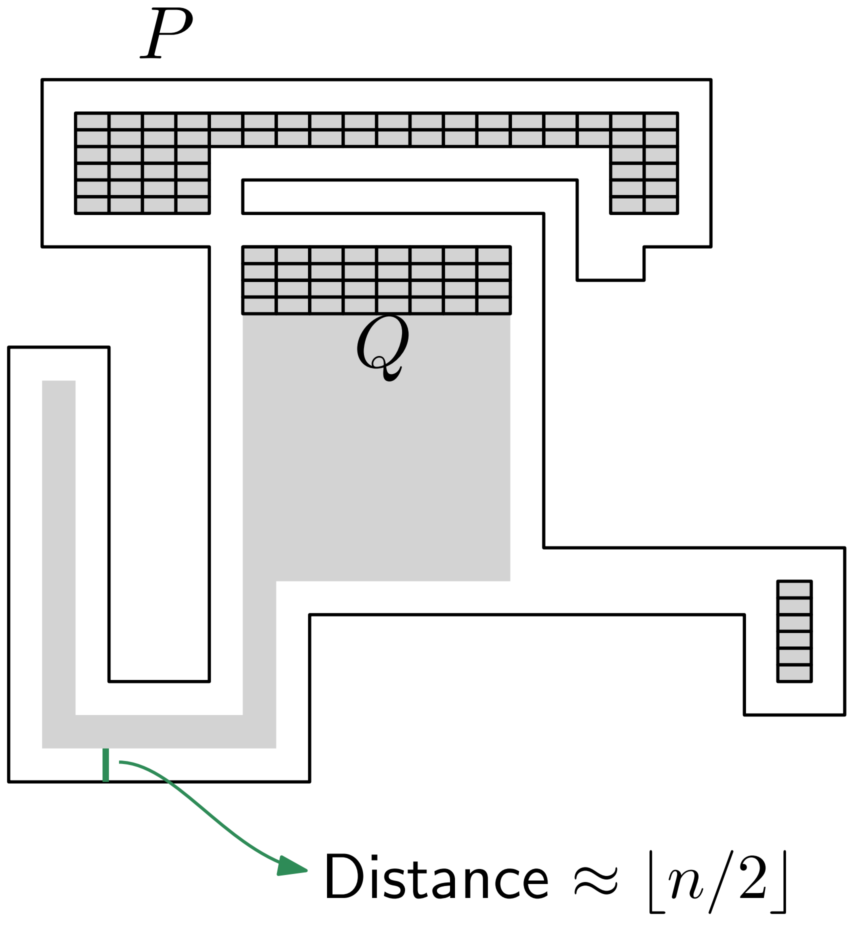

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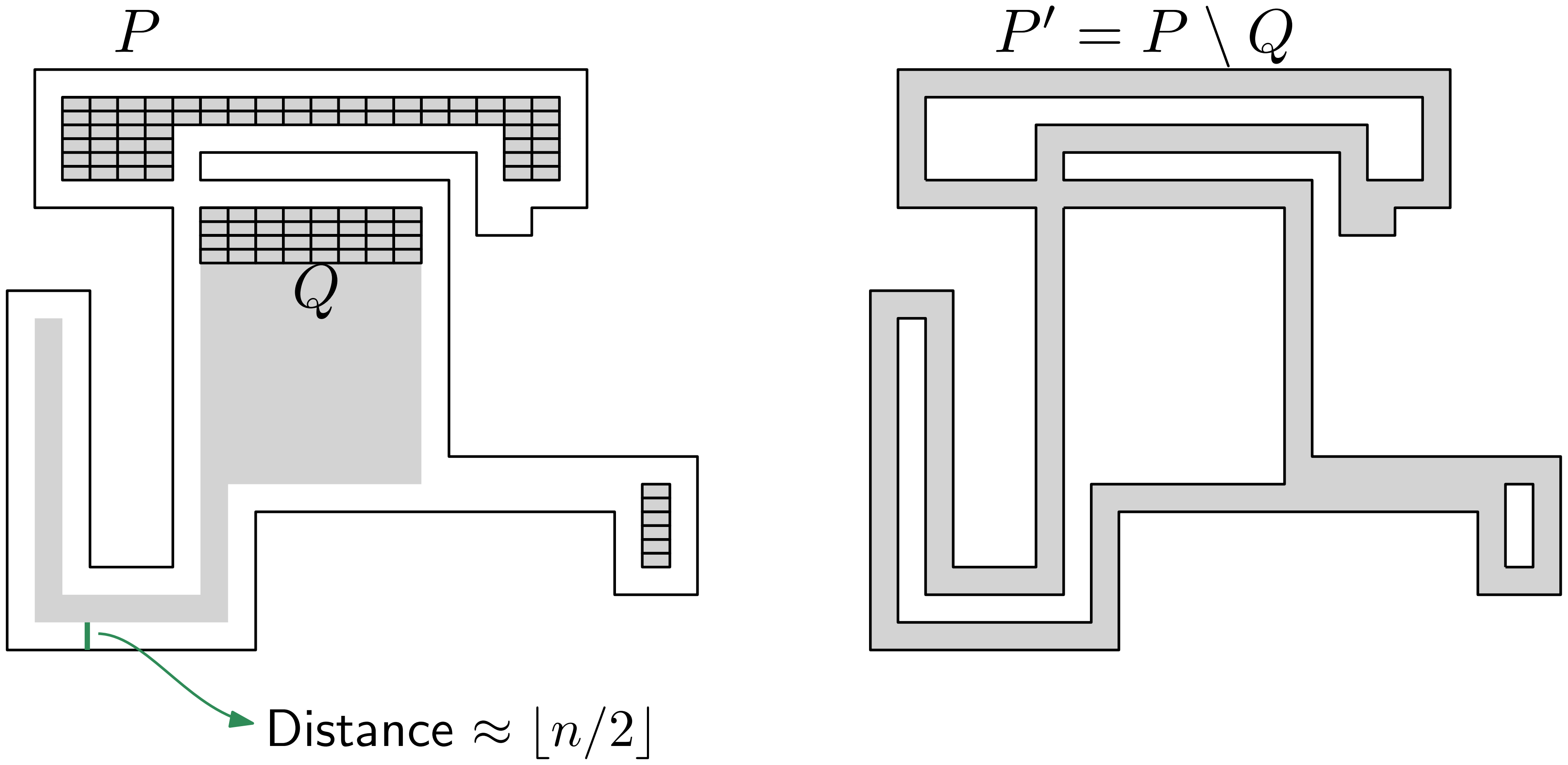

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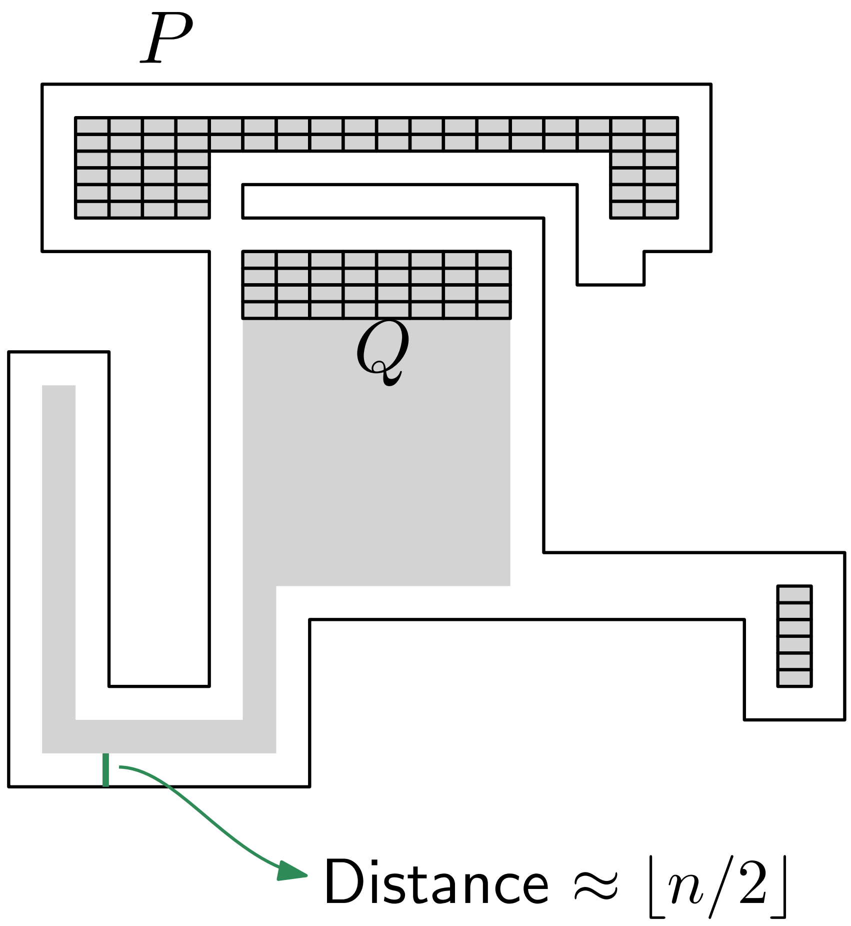
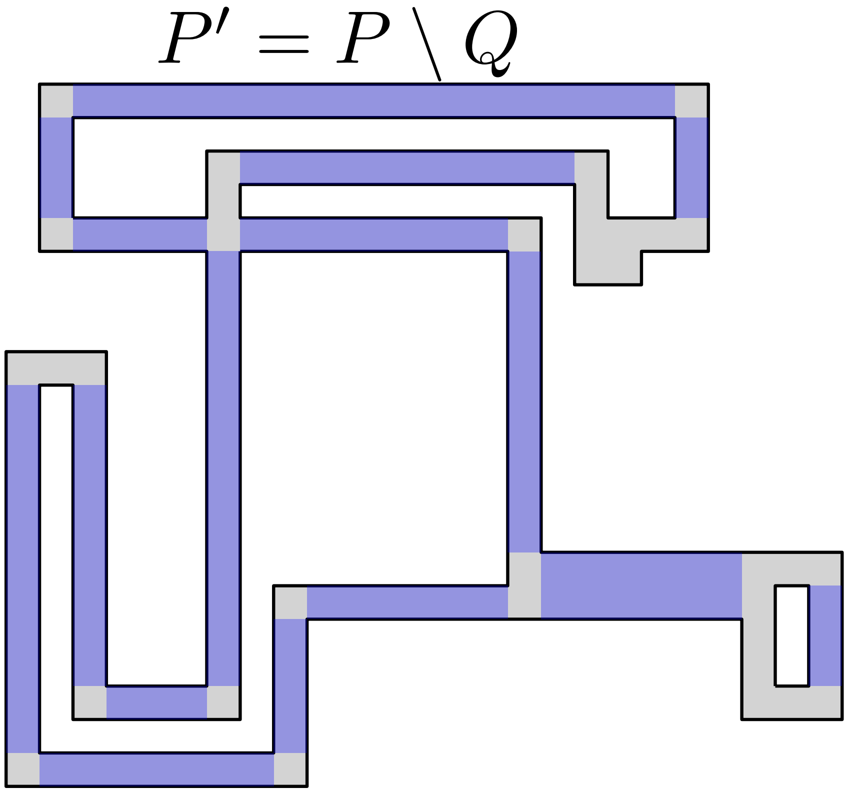
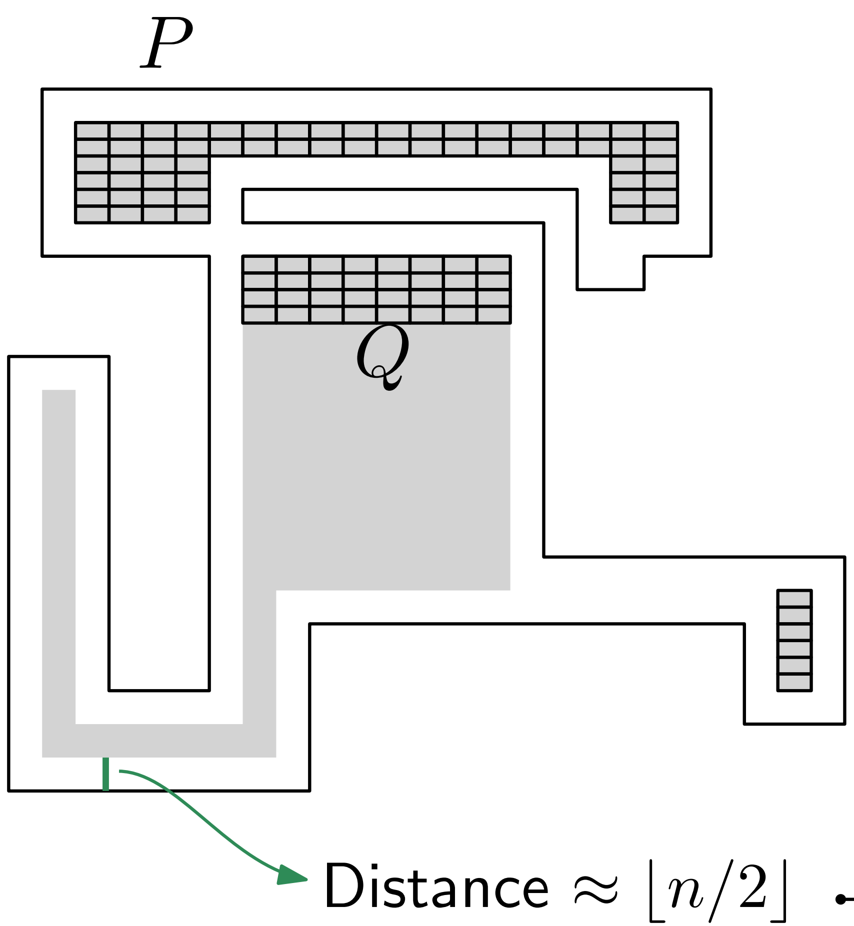
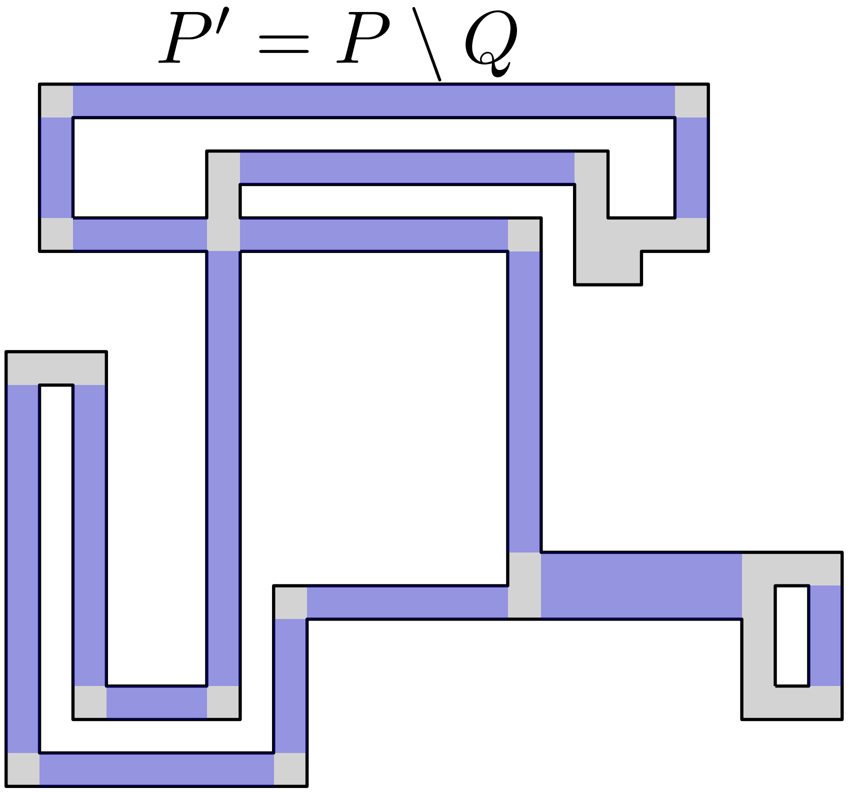


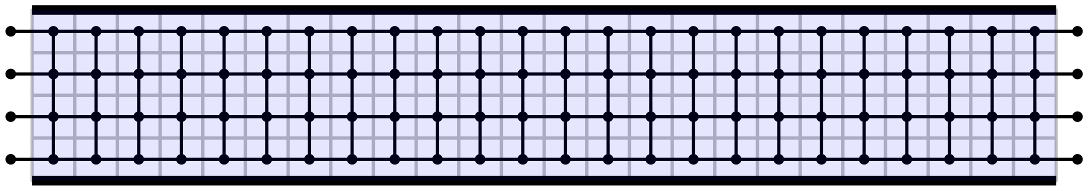


Find all *pipes* of length at least twice their width.

Cubic-time algorithm (assume no holes)



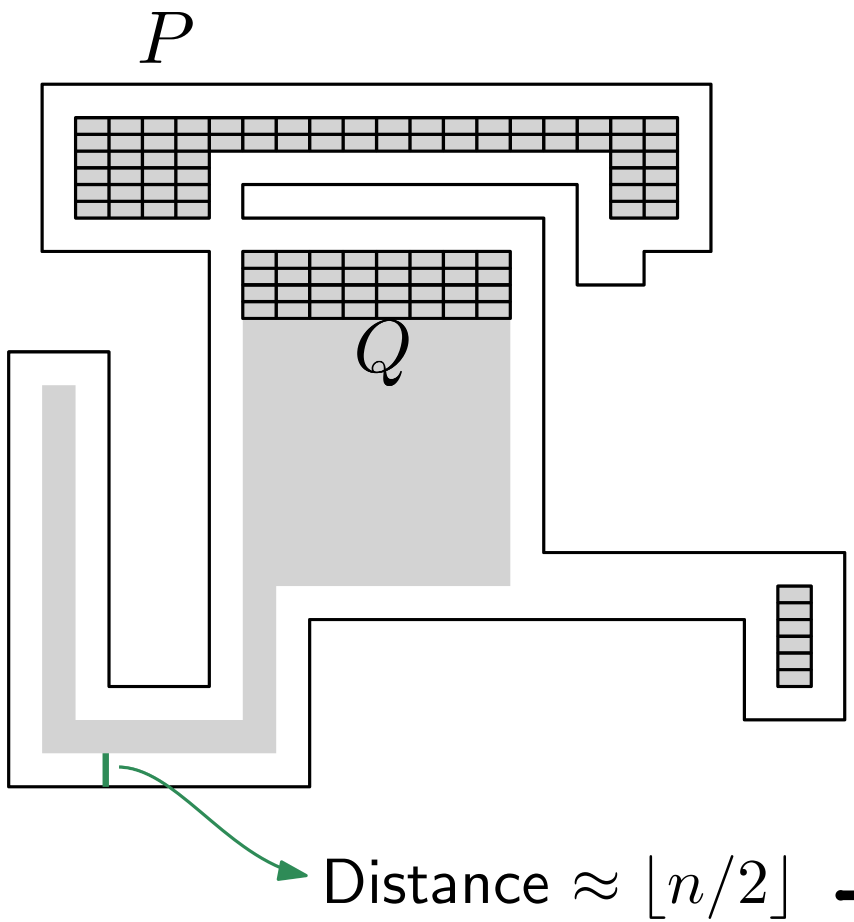


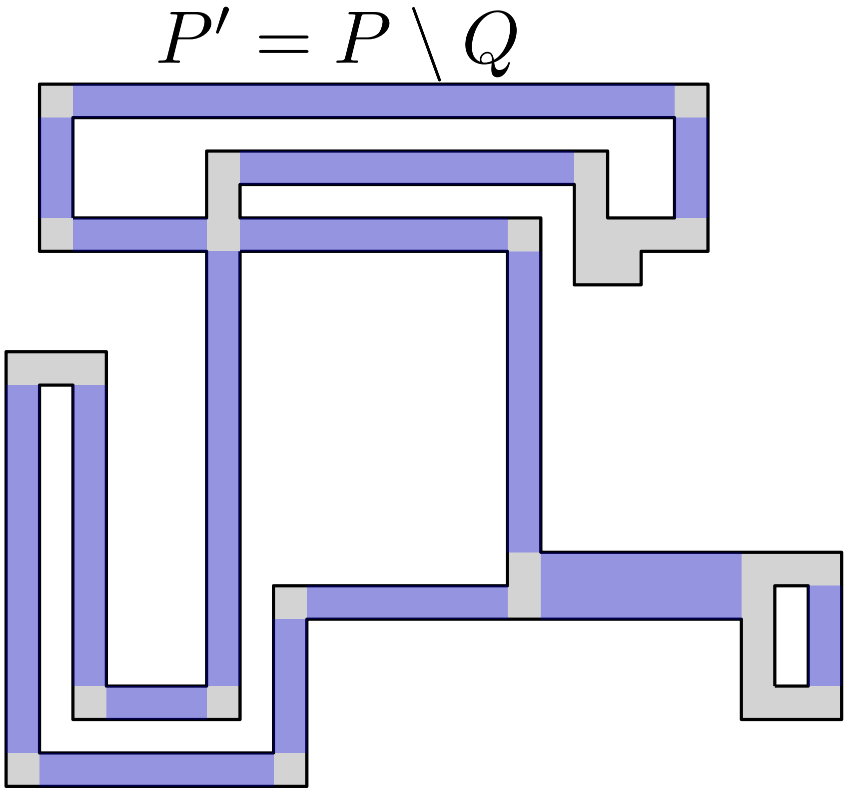


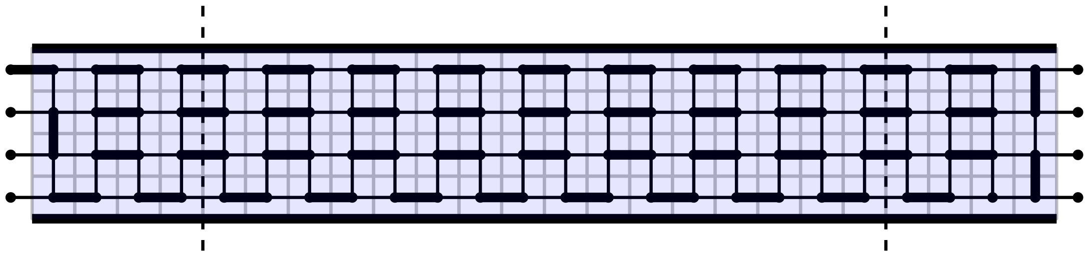
Find all *pipes* of length at least twice their width.

Shorten each pipe

Cubic-time algorithm (assume no holes)







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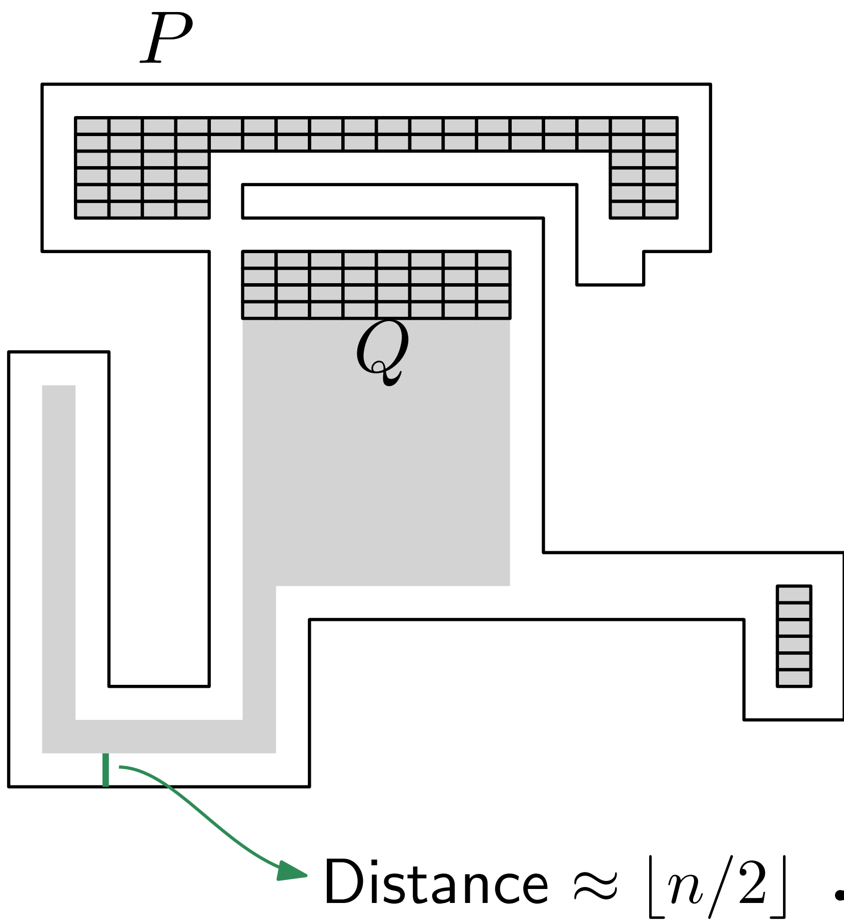

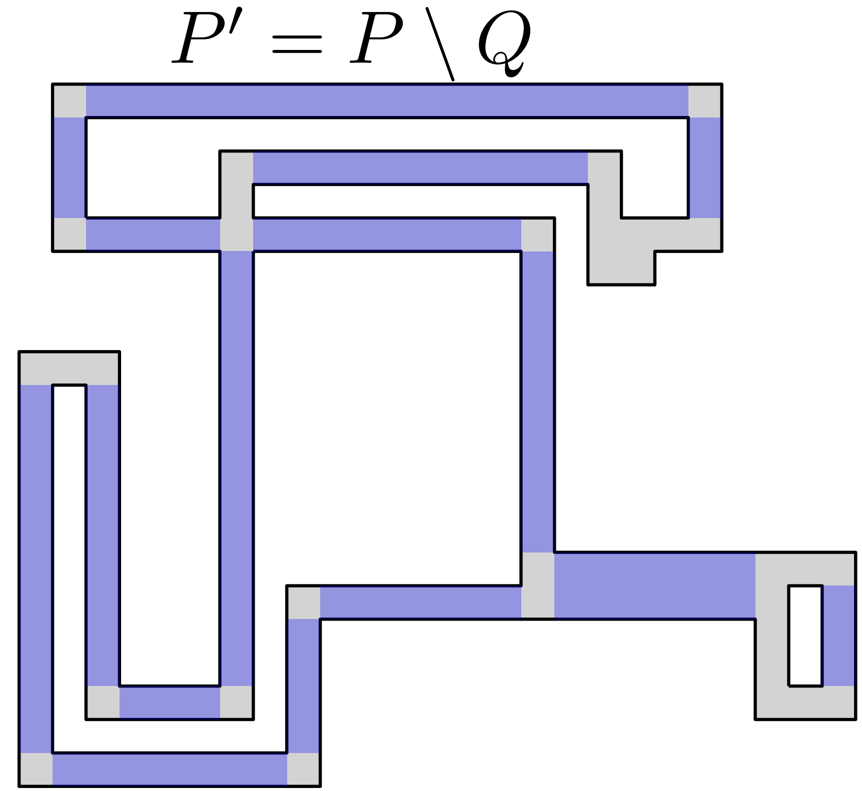

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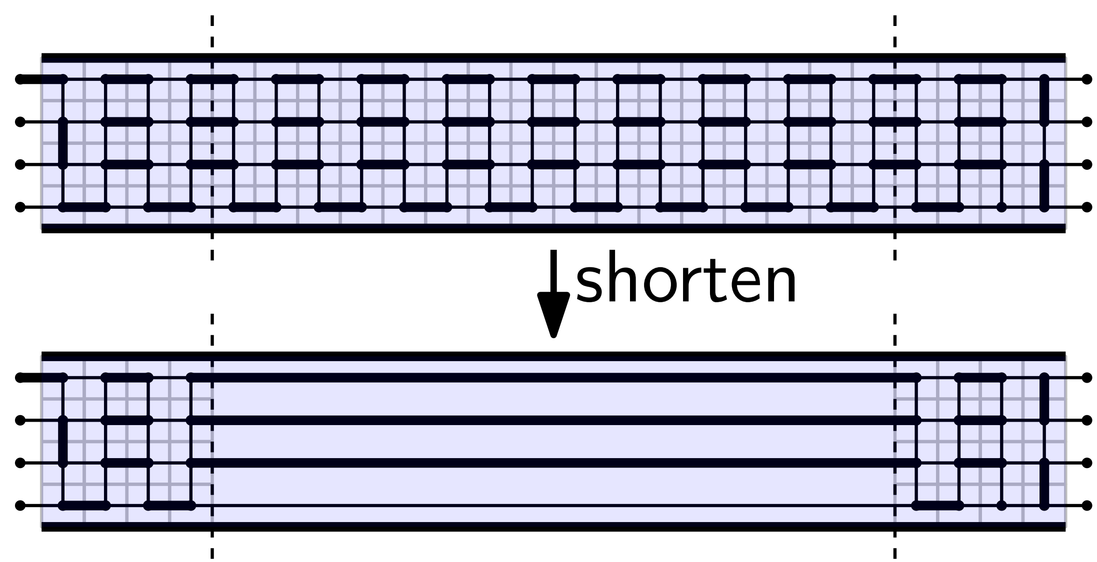

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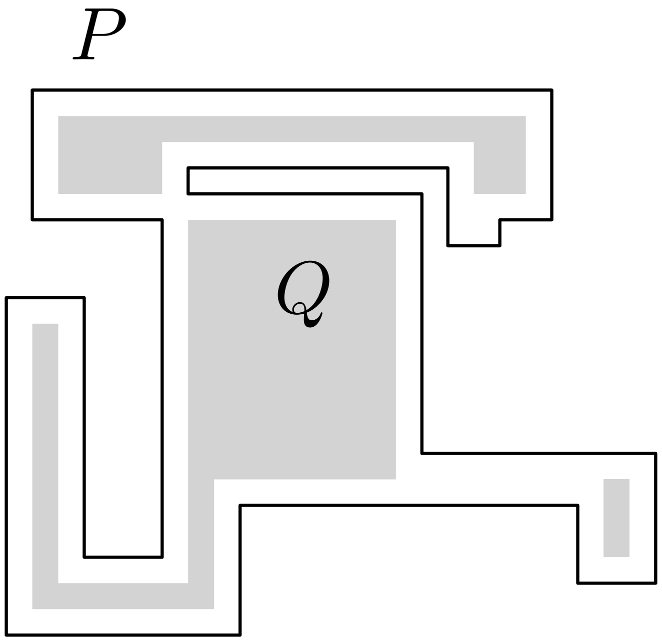

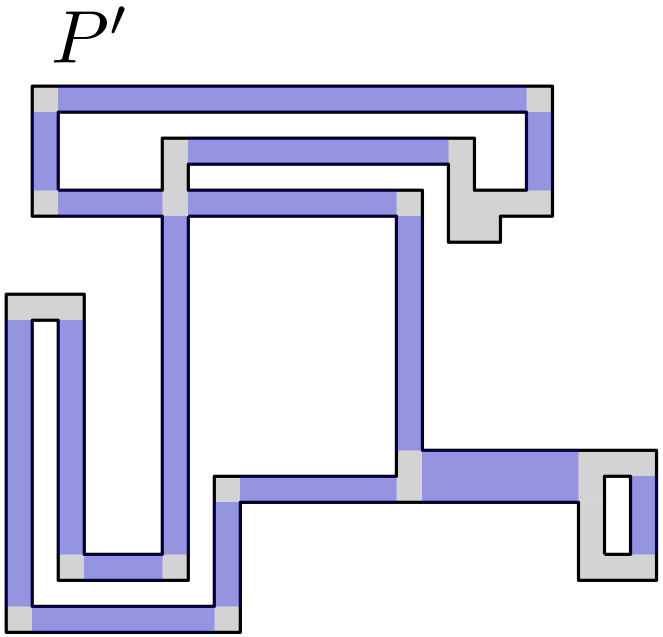

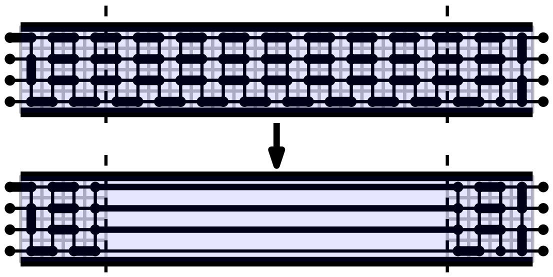

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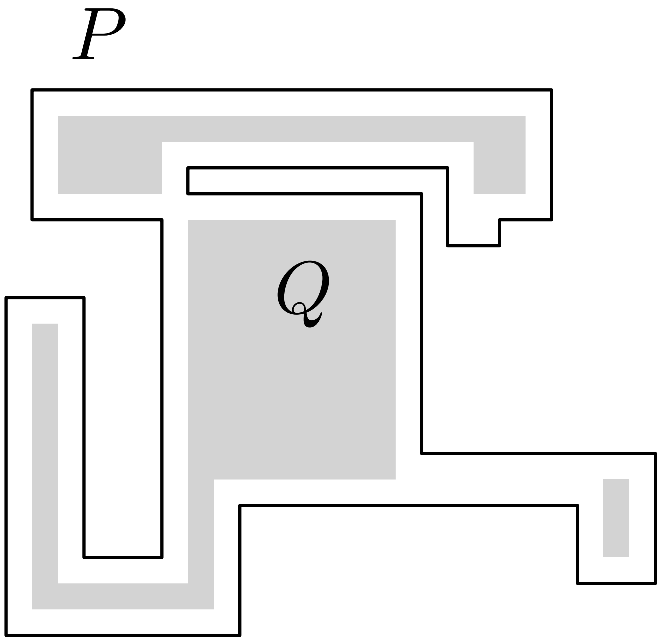

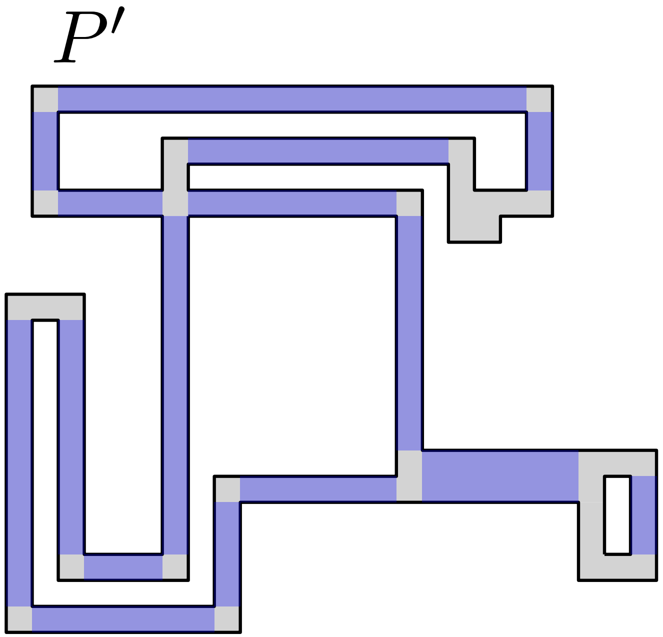

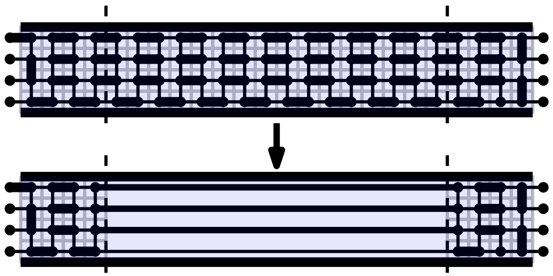

In reduced instance G^*, each vertex is of distance $O(n)$ to an original corner.

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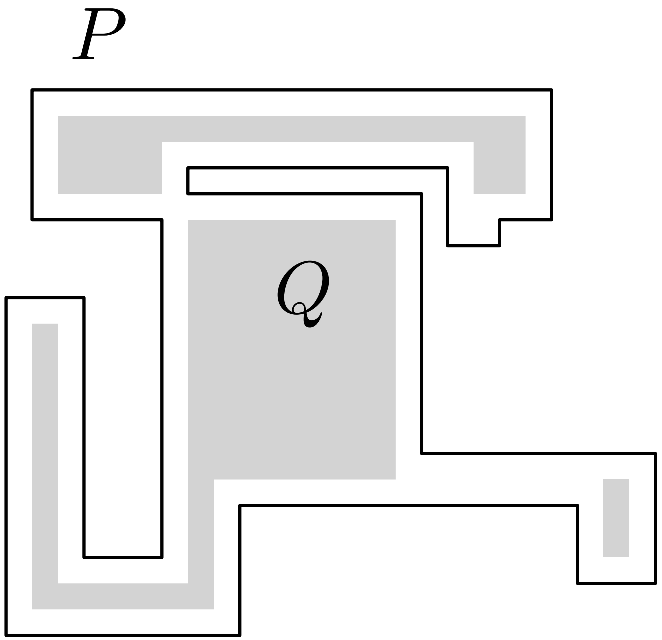

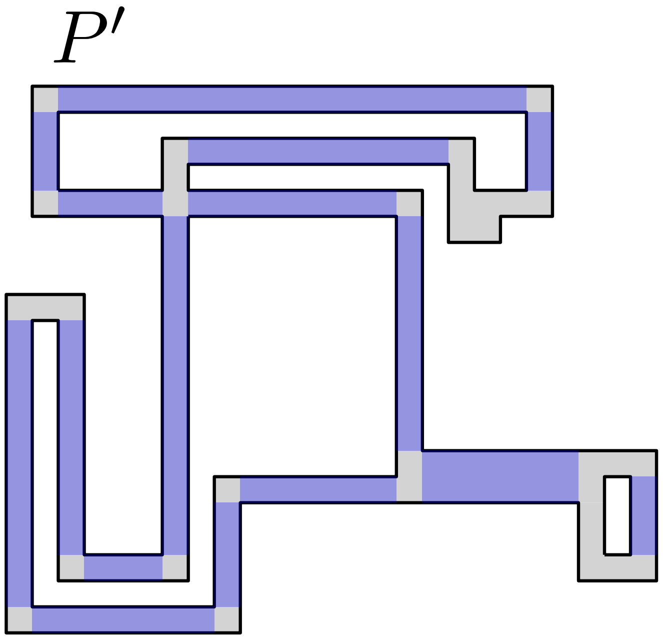

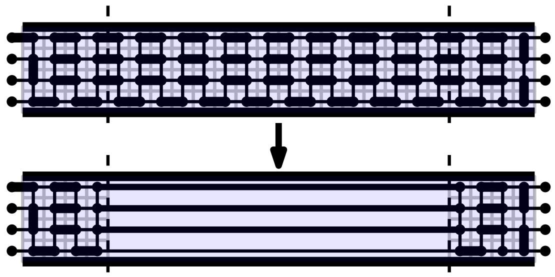

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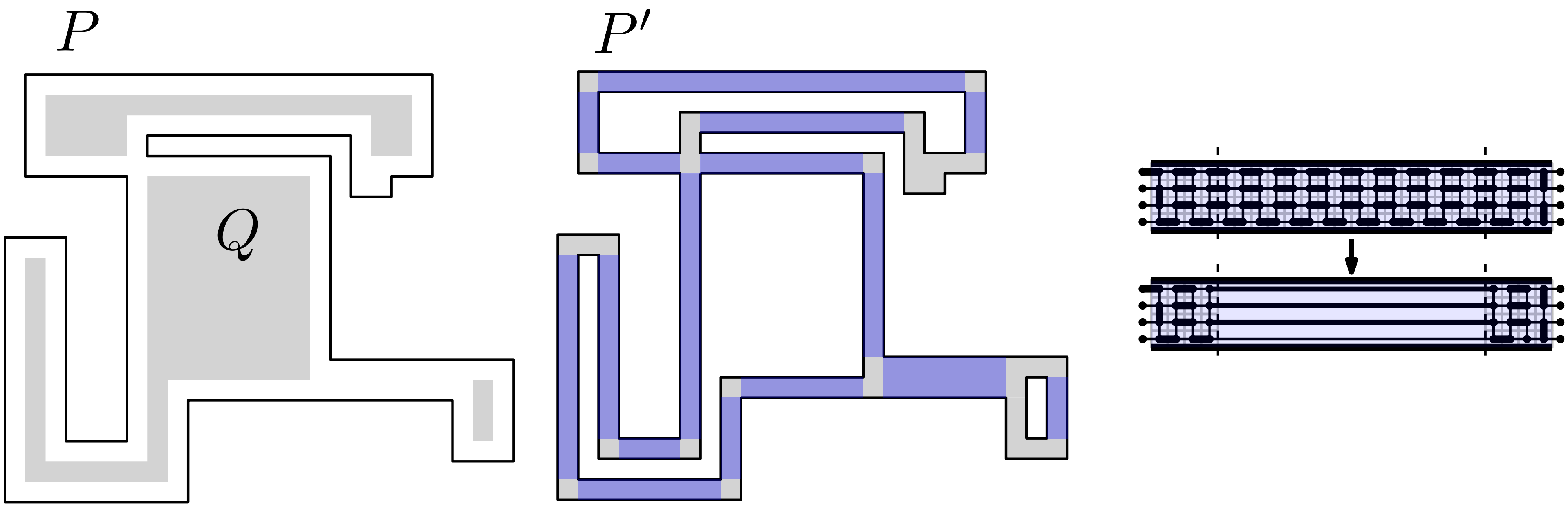

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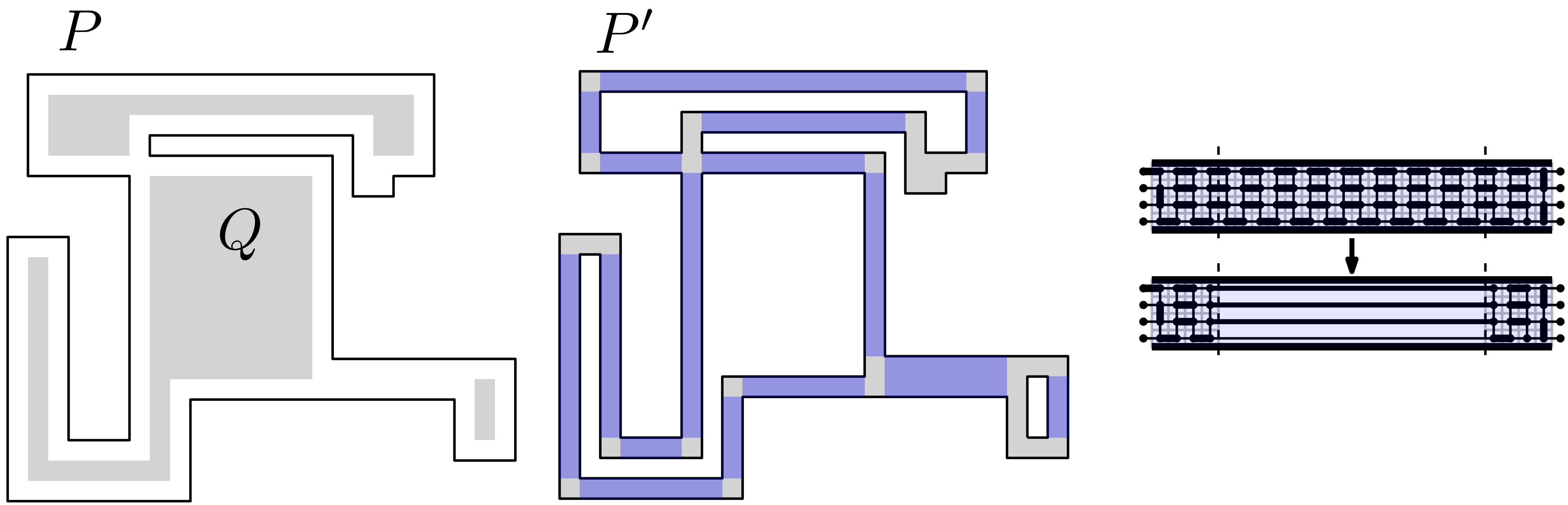

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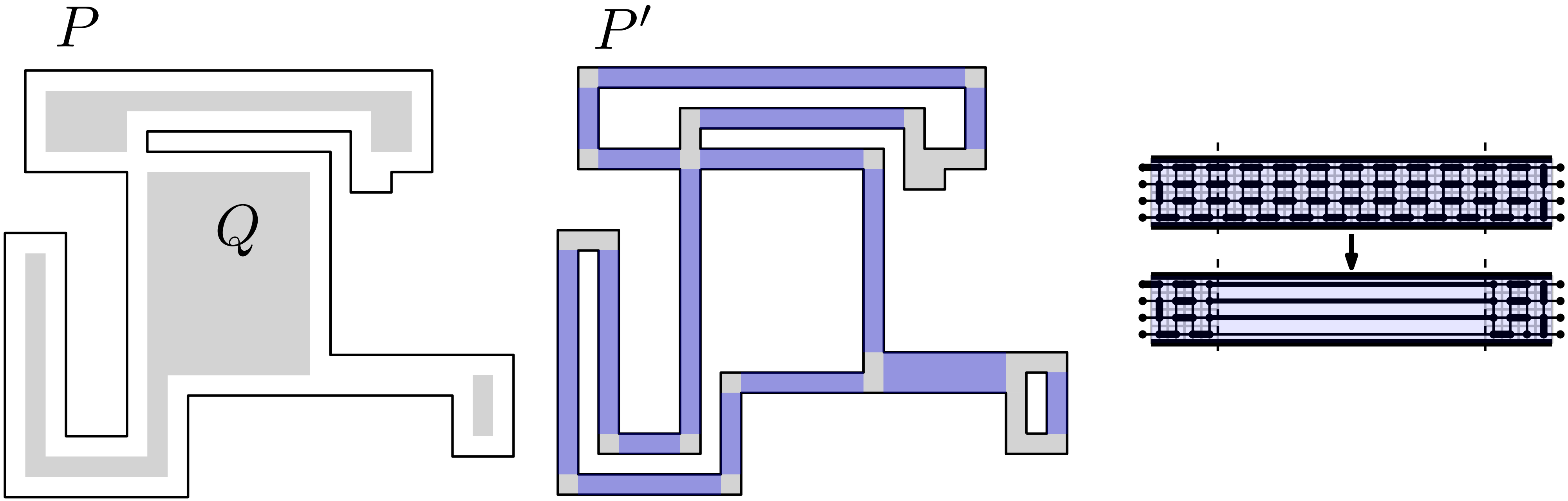

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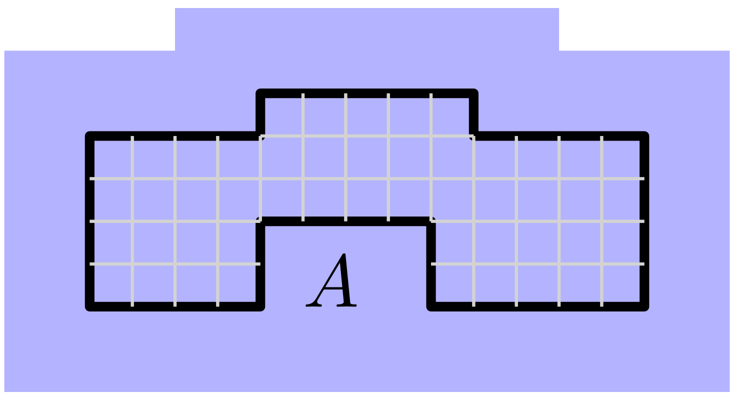
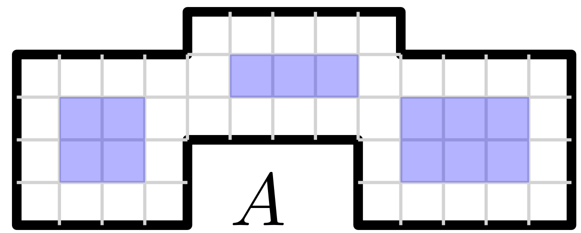
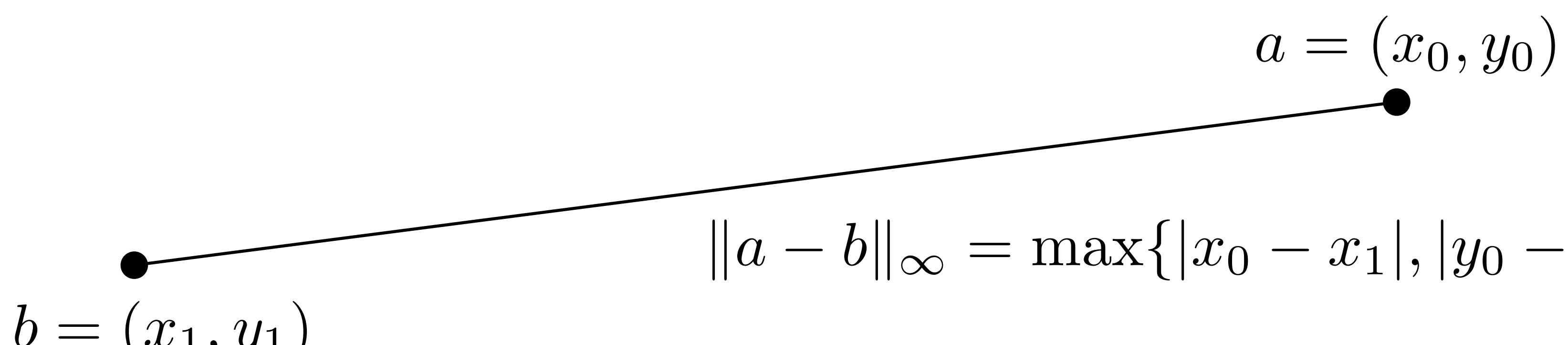
Return $|M| + \frac{\text{area}(P) - V(G^*)}{2}$.

Offsets

$B(A, r)$: Offset of A by distance r wrt. $\| \cdot \|_\infty$.



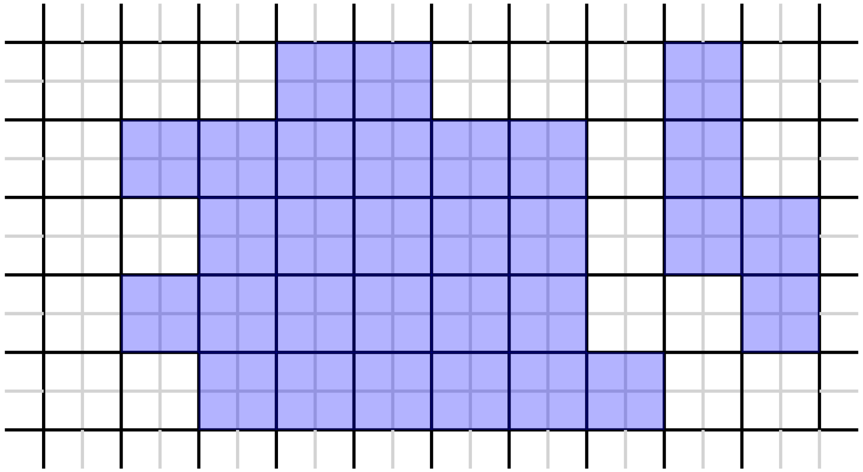




$$\|a - b\|_\infty = \max\{|x_0 - x_1|, |y_0 - y_1|\}$$

Consistent Parity

A polyomino $P \subset \mathbf{R}^2$ has **consistent parity** if all first coordinates of corners of P have the same parity and vice versa for the second coordinates



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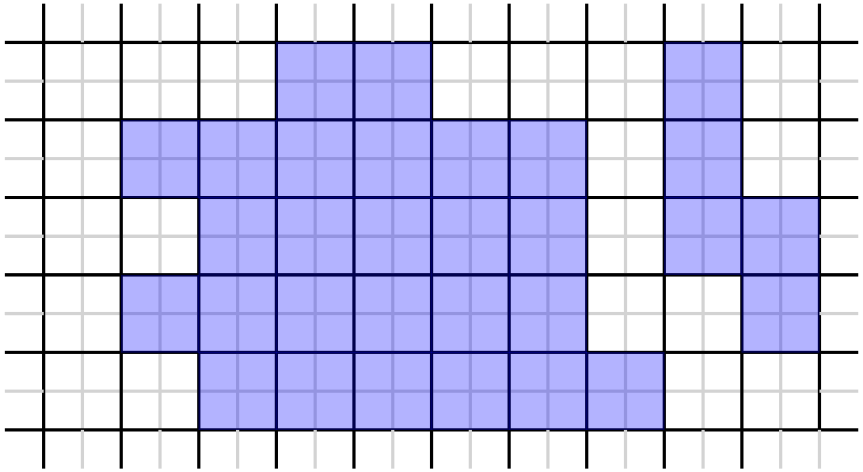
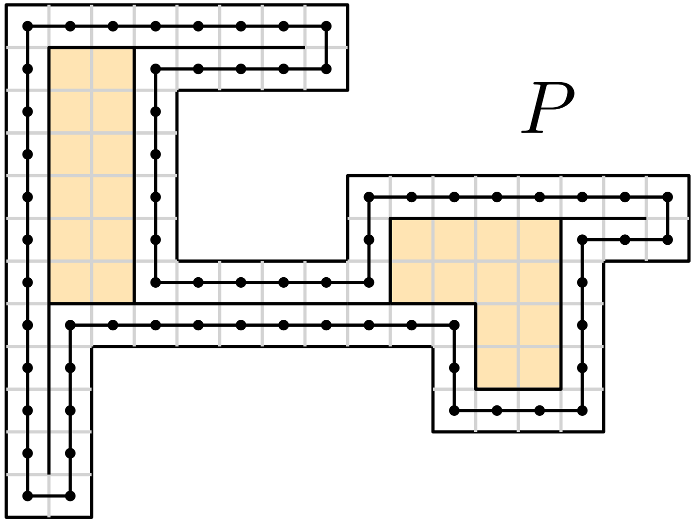


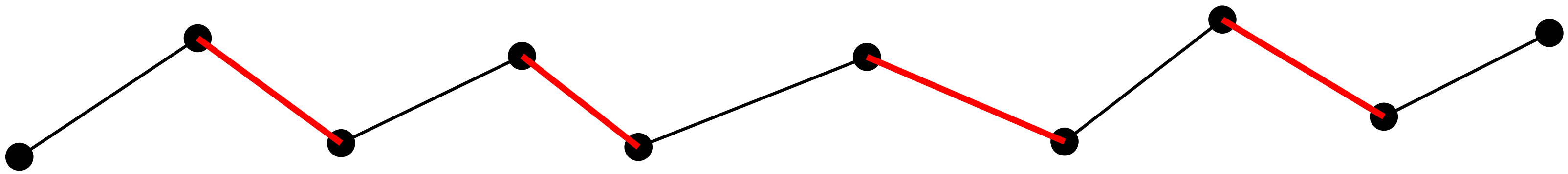
Observation: *If P has no holes and consistent parity then each component of $P \setminus B(P, -1)$ is Hamiltonian.*



Augmenting paths

Let G be a graph, M a matching of G.

A path $P = v_1, v_2, \ldots, v_{2k}$ of G is **augmenting** if v_1 and v_{2k} are unmatched and $(v_{2i}, v_{2i+1}) \in M$, $i = 1, \ldots, k-1$



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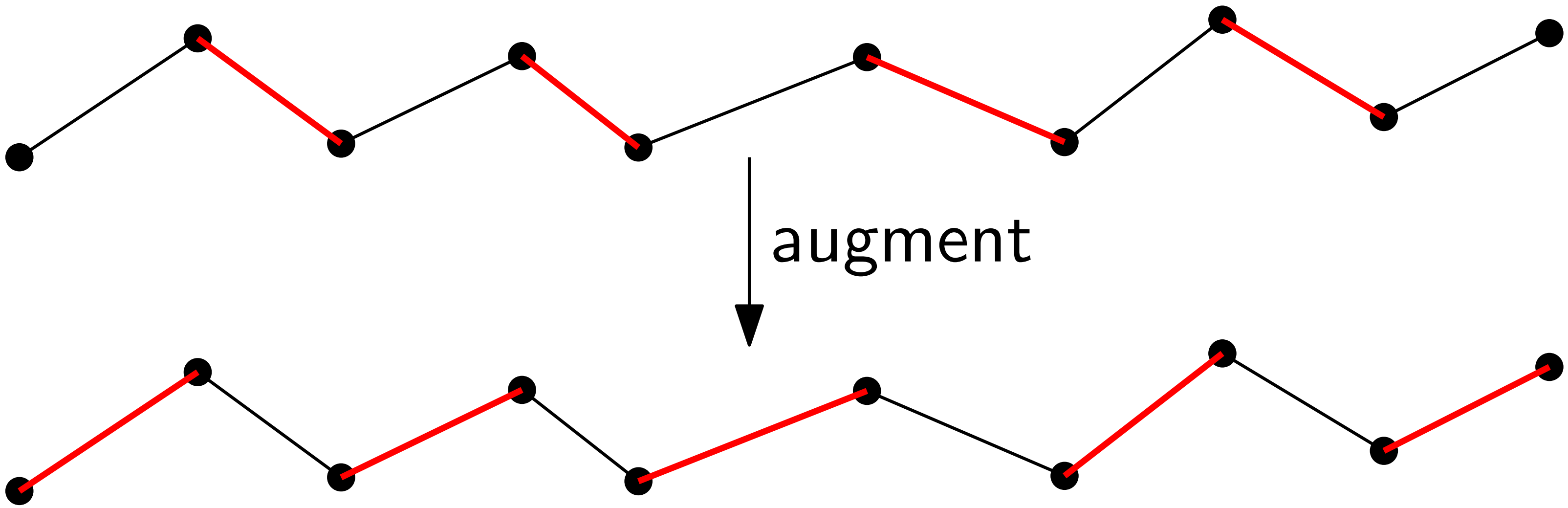

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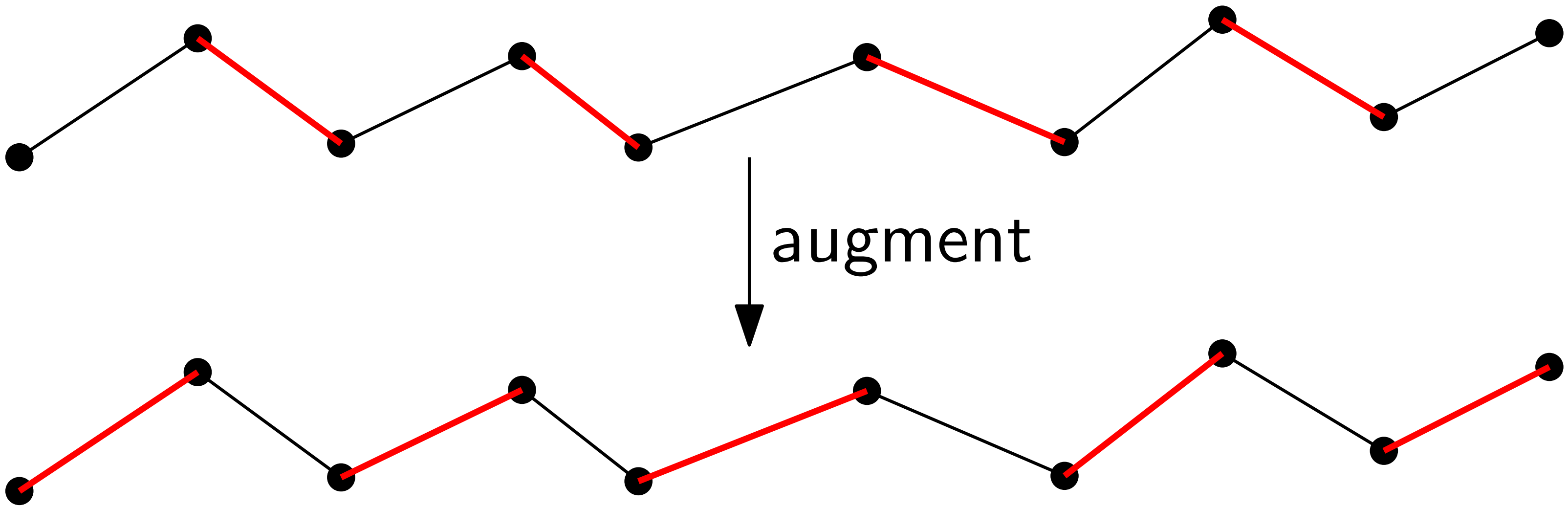

Lemma (Berge). *Let G be a graph and M a matching of G which is not maximum. Then there exists an augmenting path between two unmatched vertices of G.*

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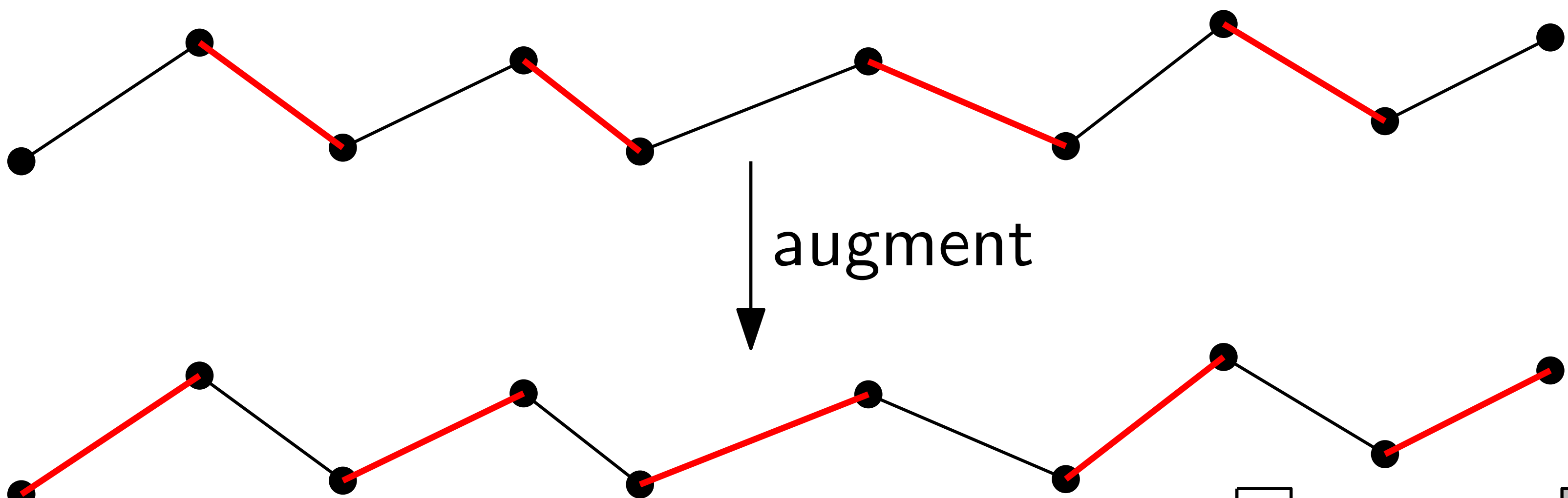

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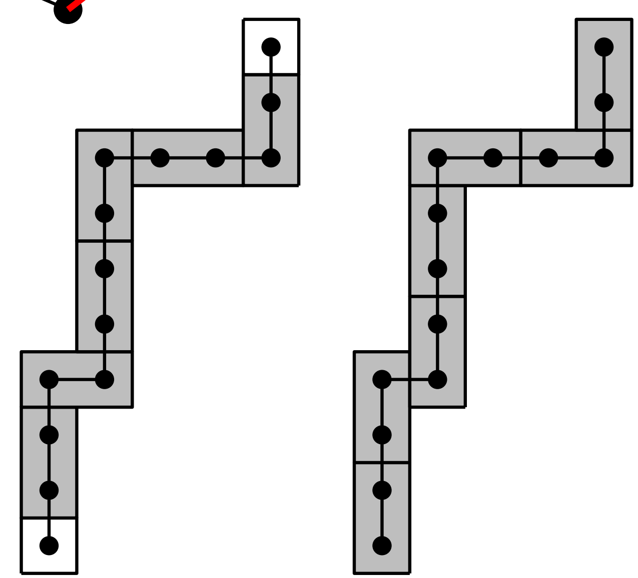
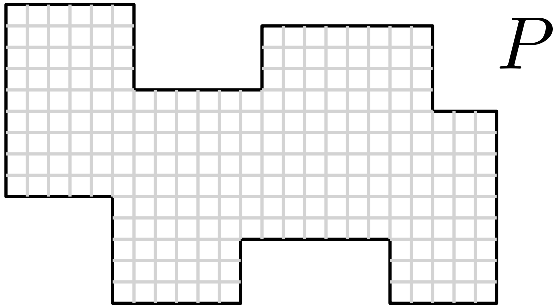


Structural Result 1

Let subpolyomino $P_0 \subseteq P$ be maximal with consistent parity.



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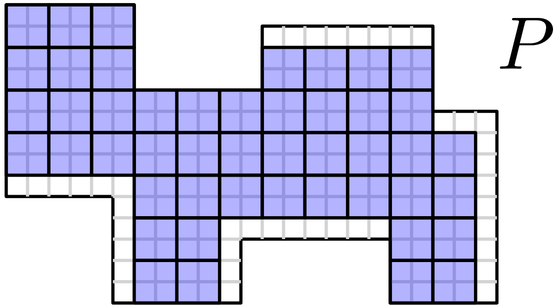

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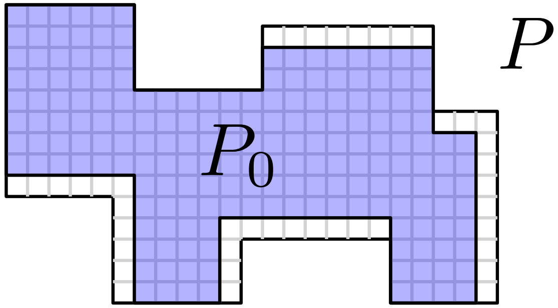

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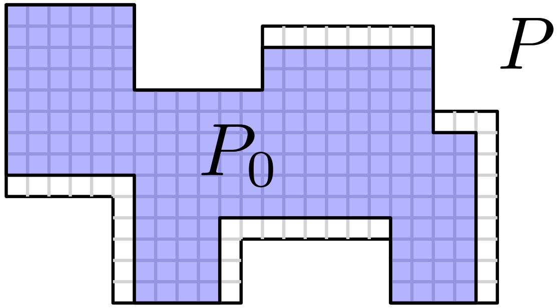

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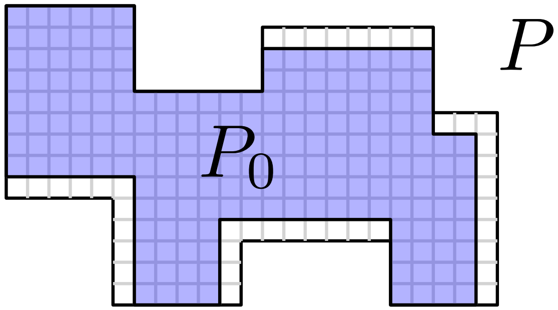

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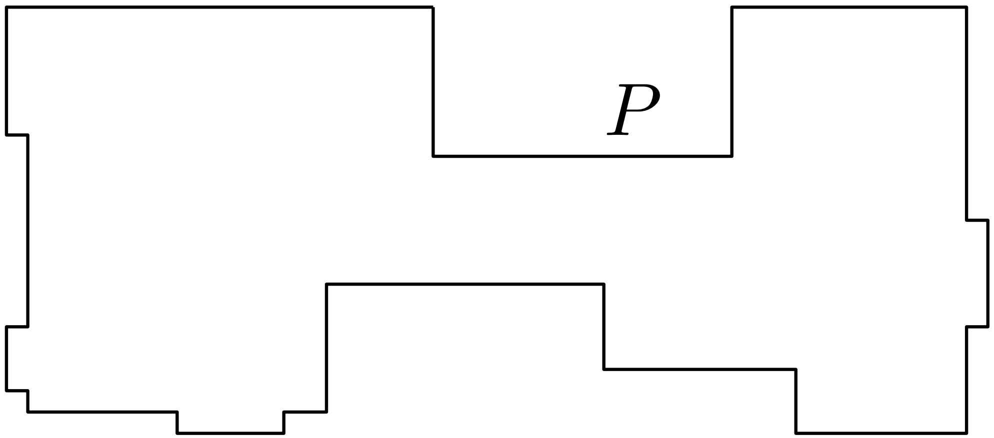

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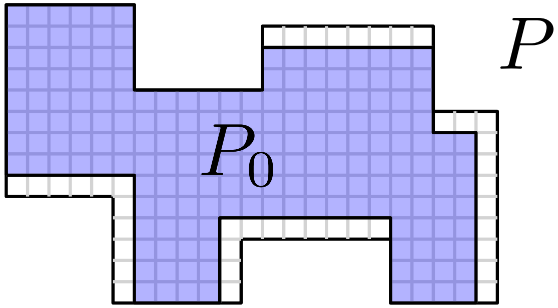

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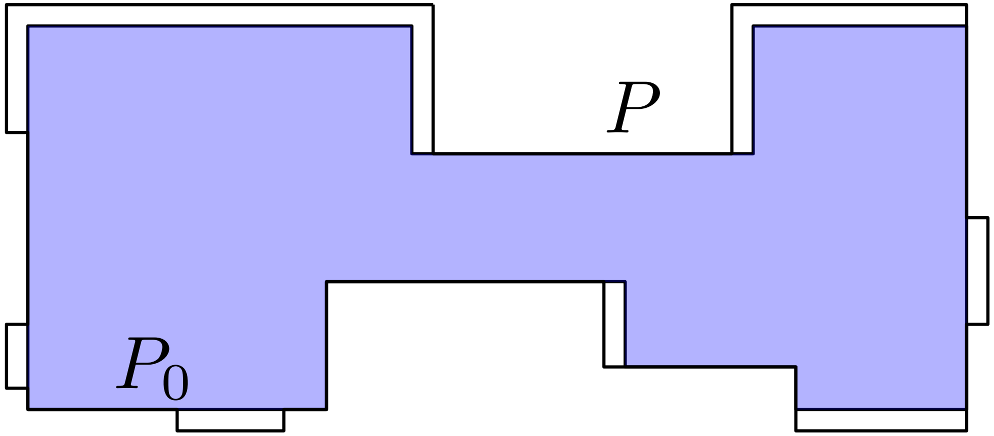

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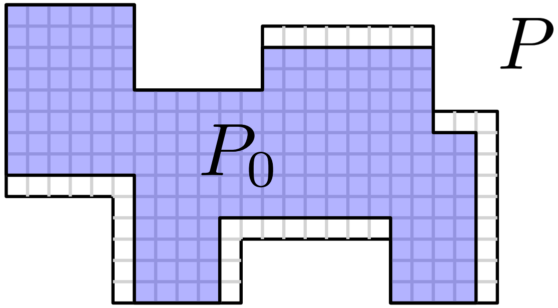

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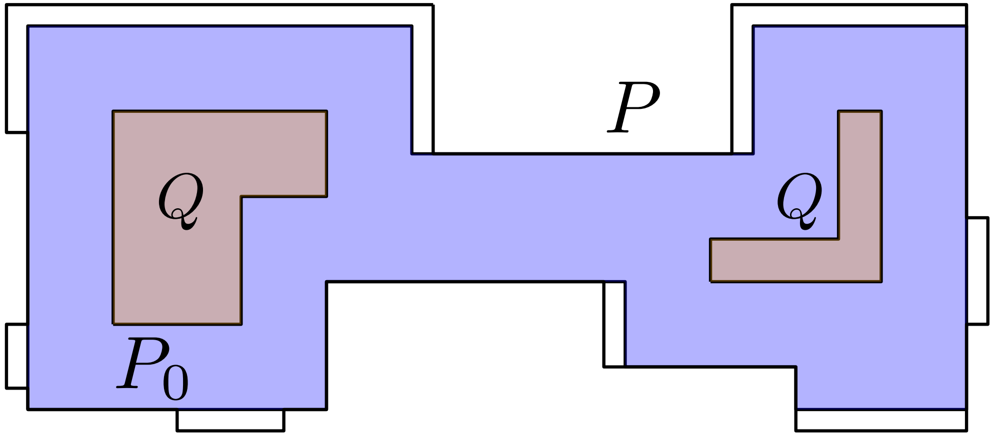

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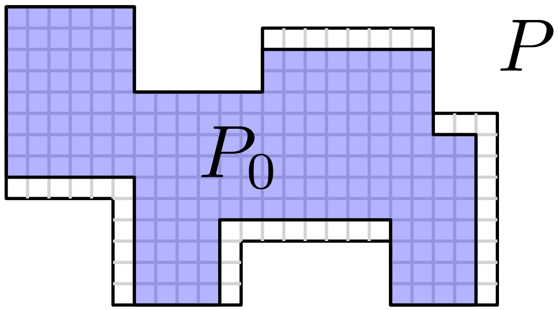

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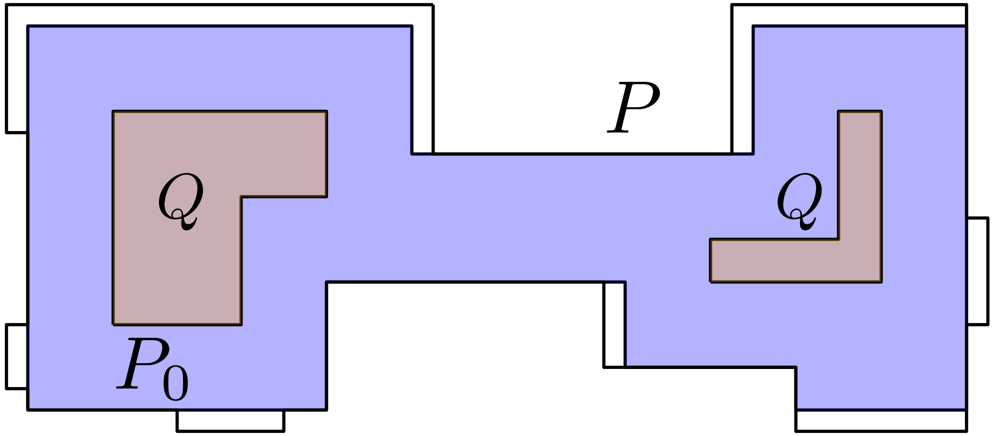

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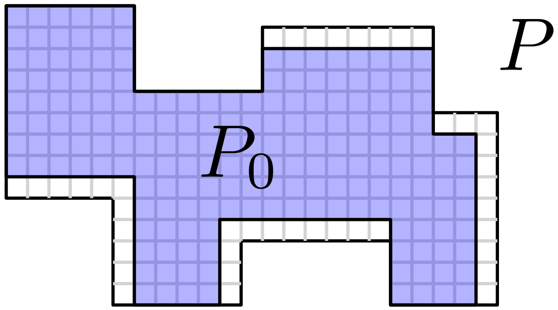

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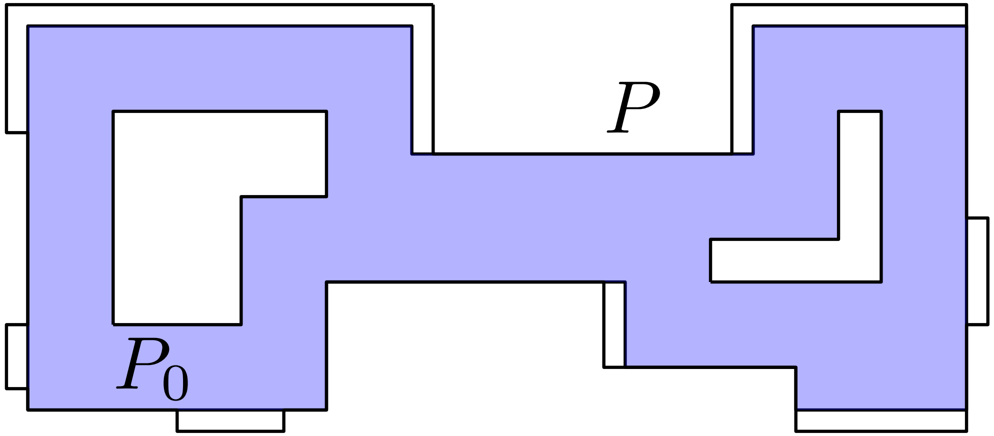

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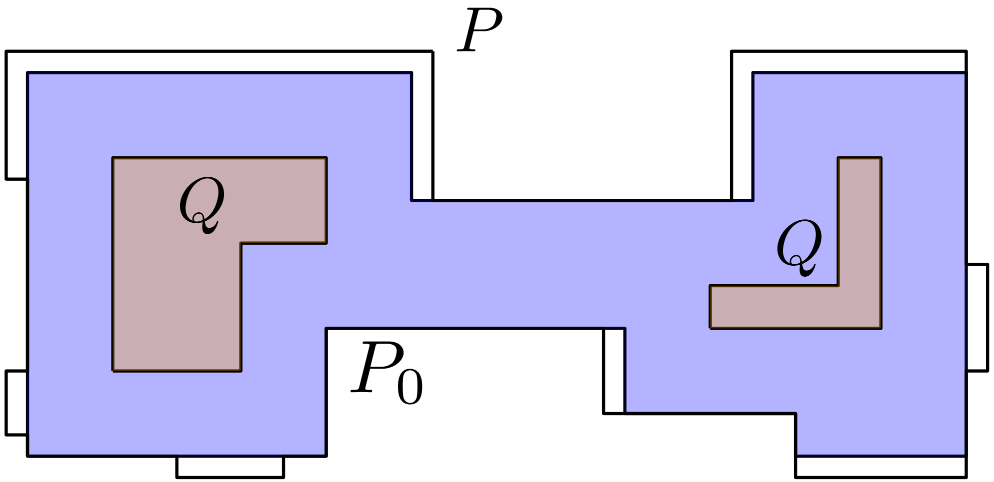

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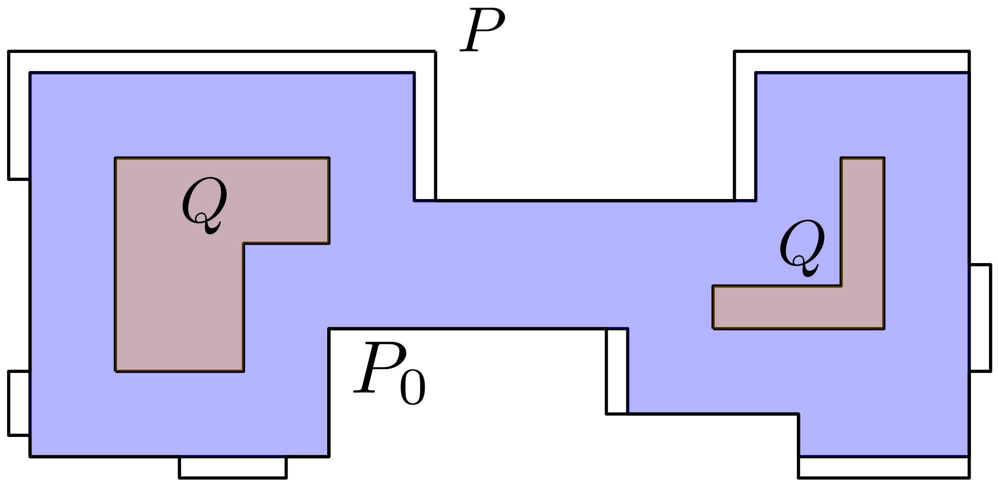

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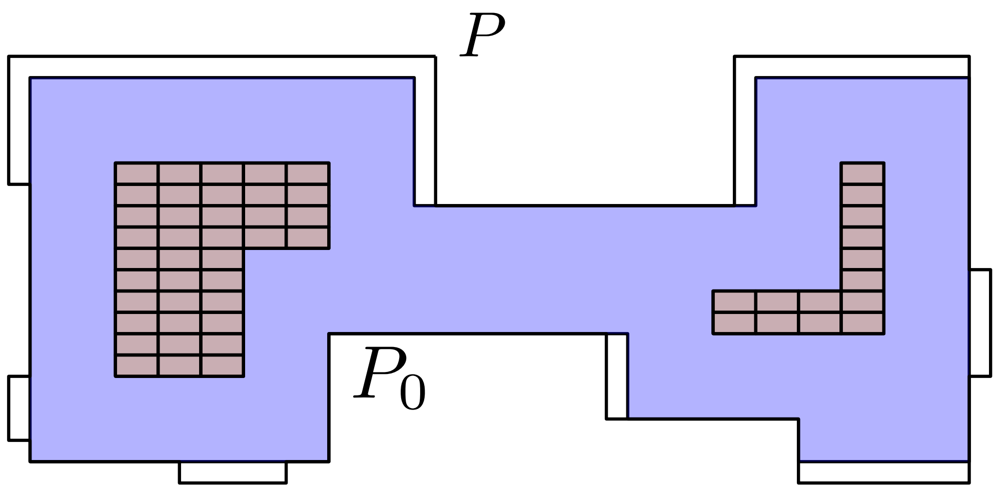

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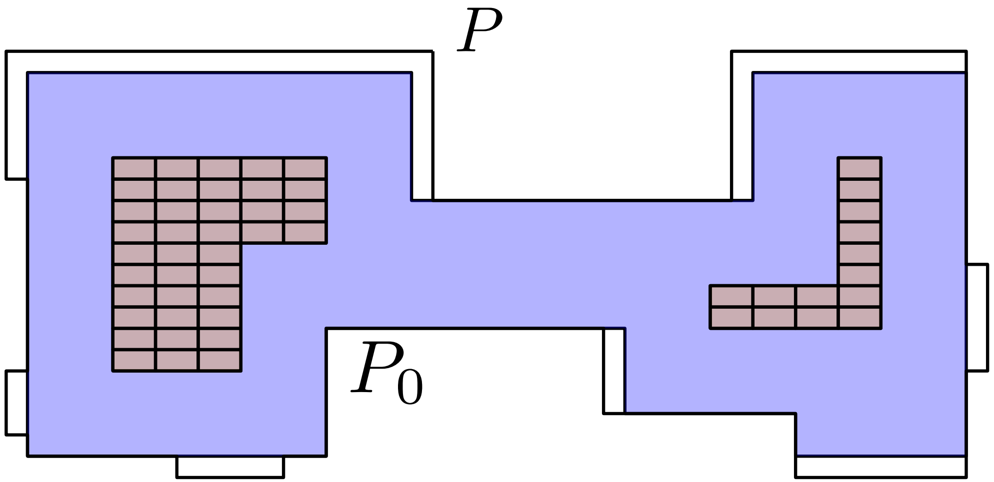

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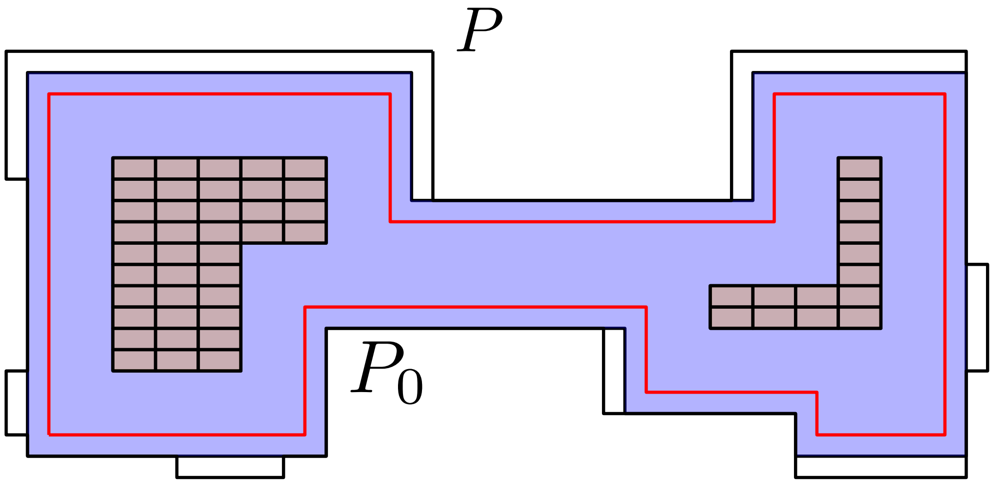

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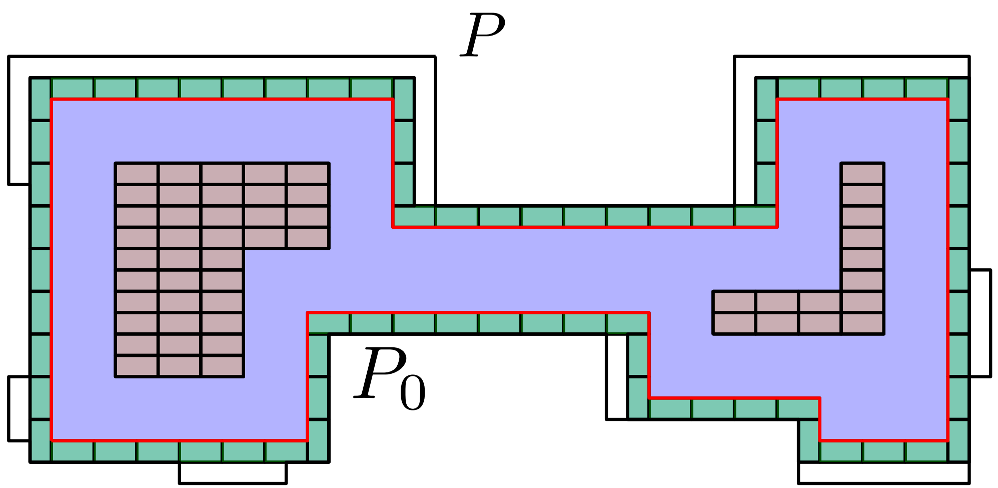

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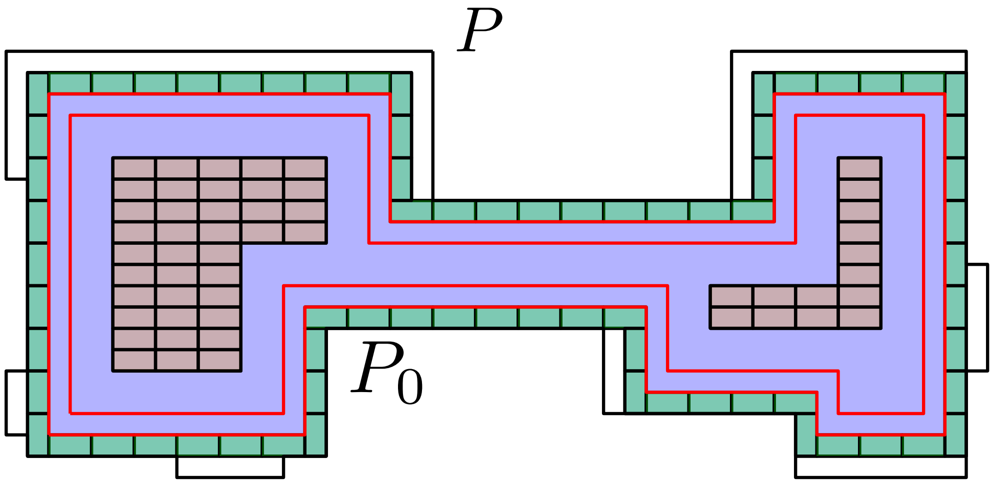

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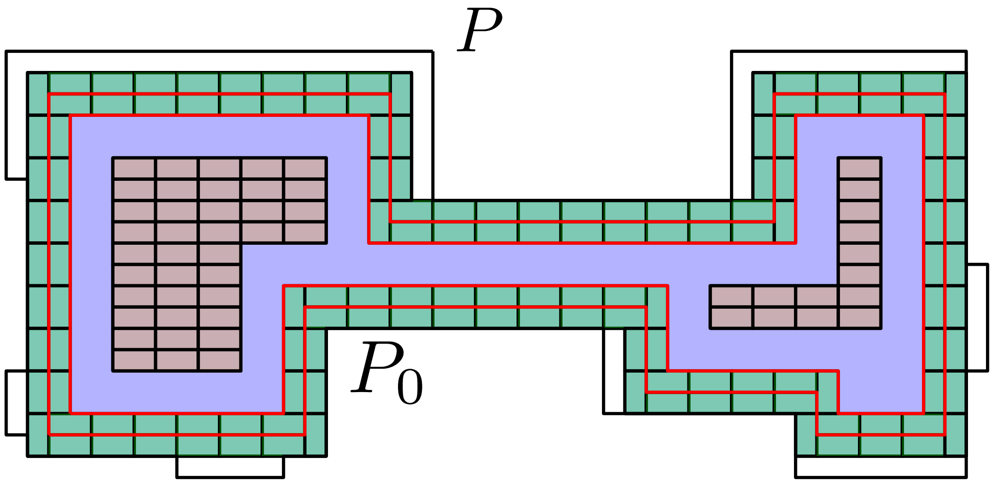

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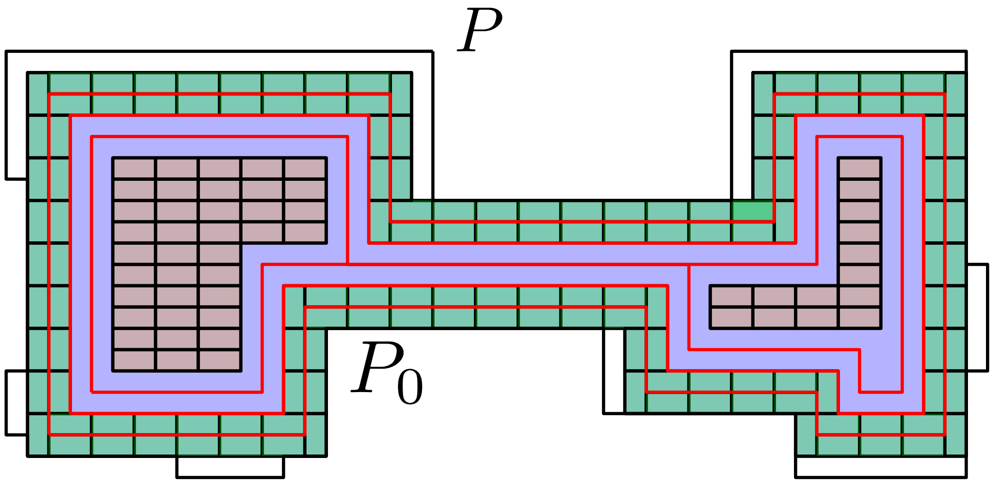

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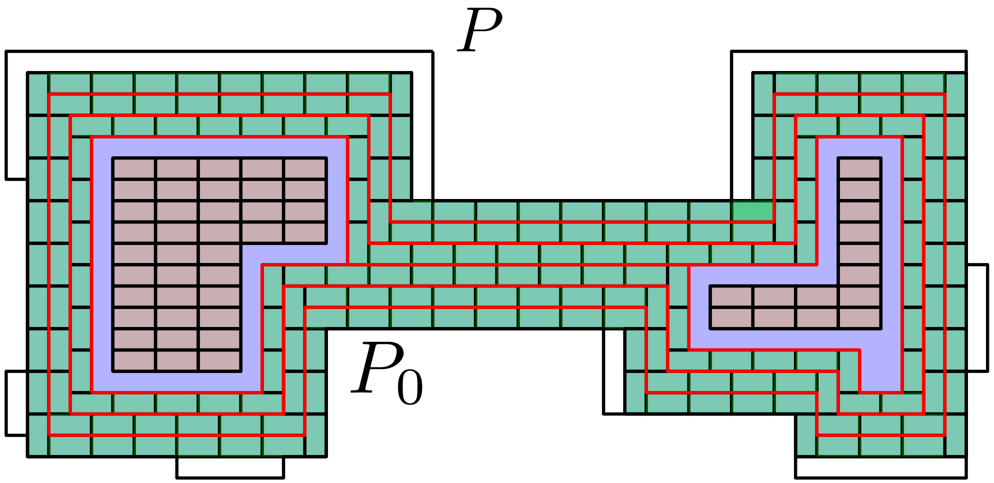

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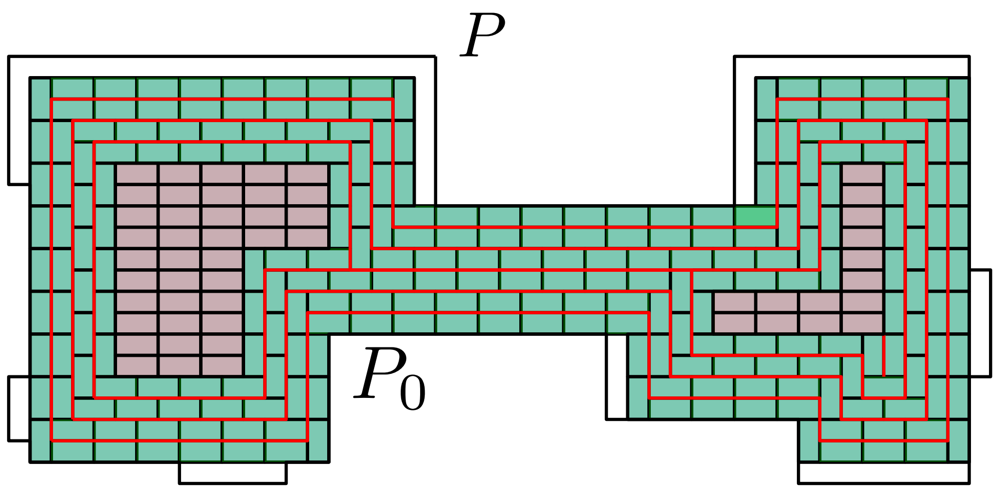

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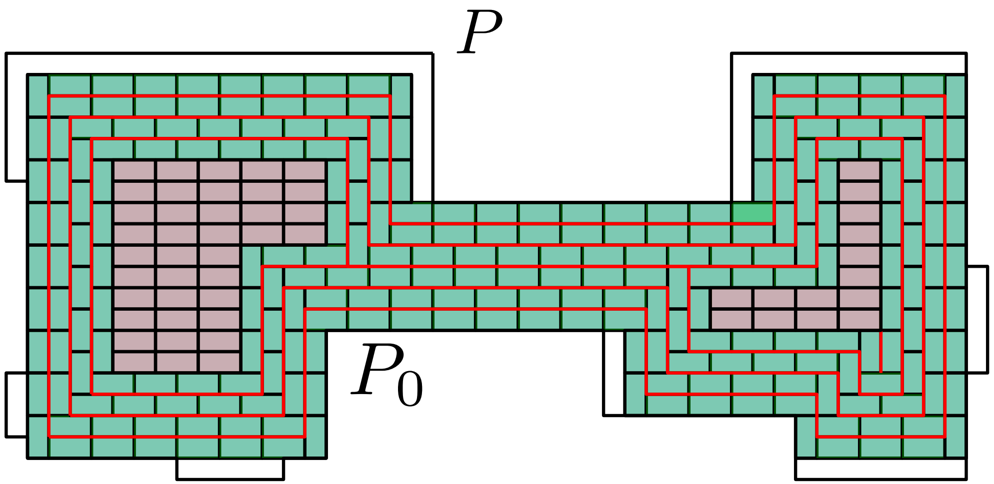

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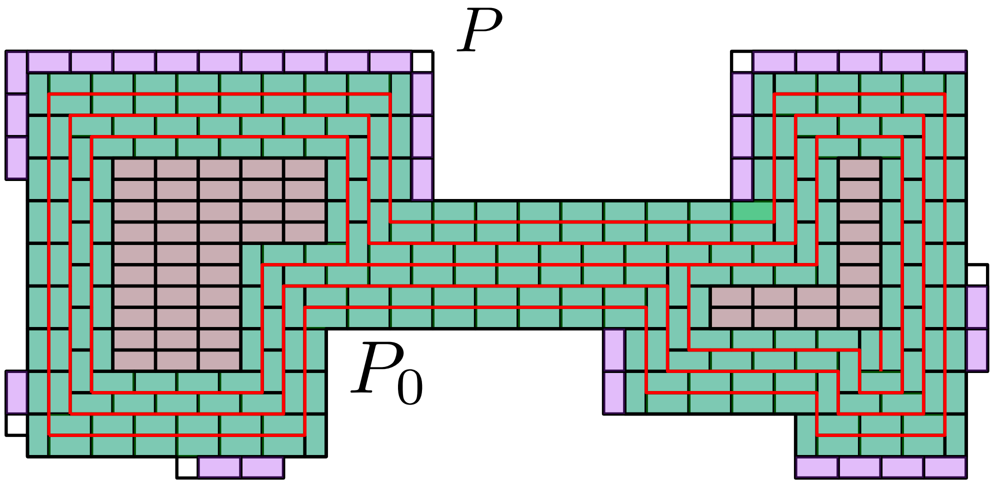

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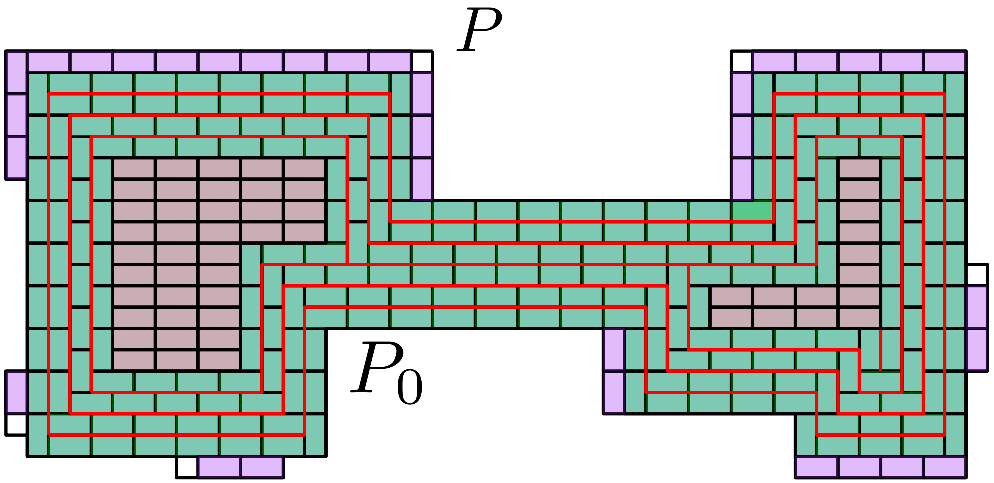

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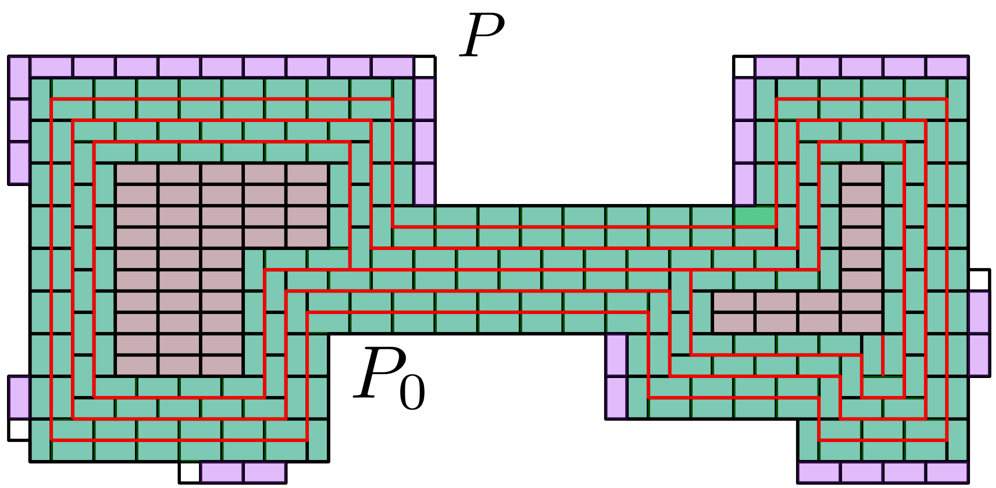

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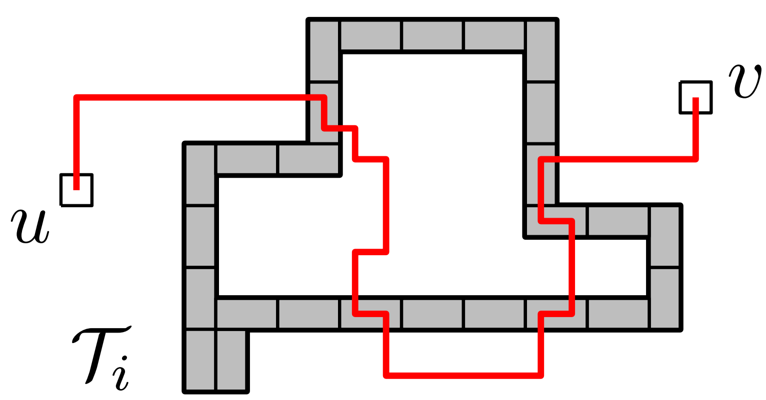

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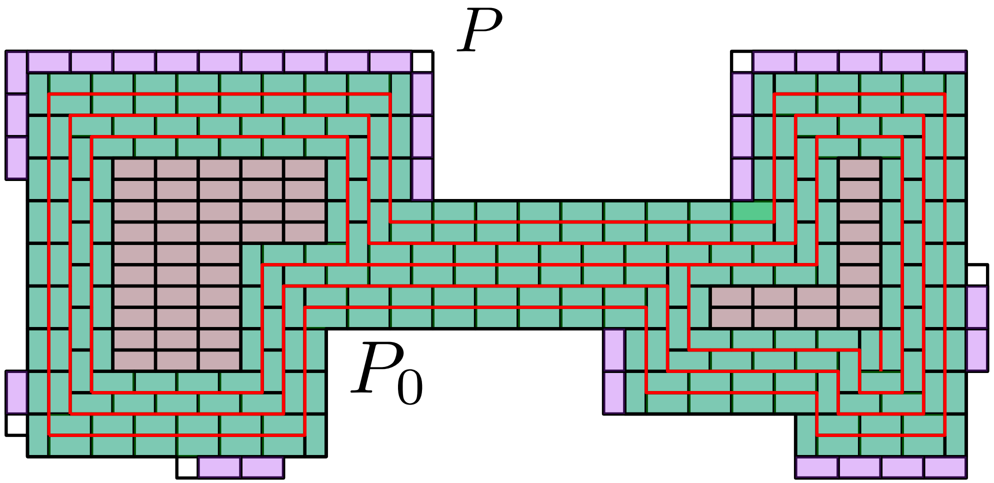

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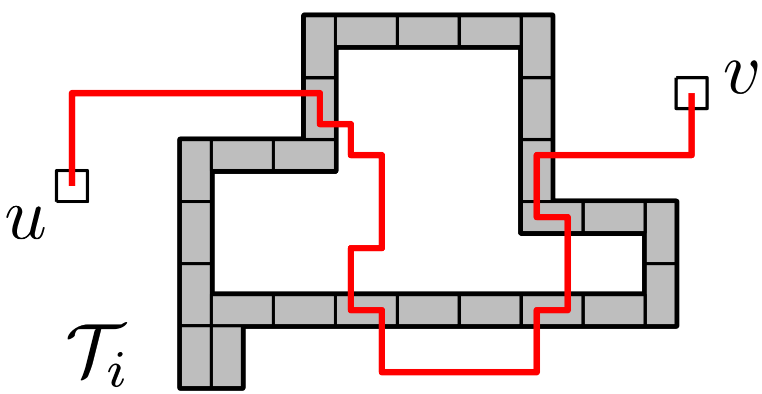

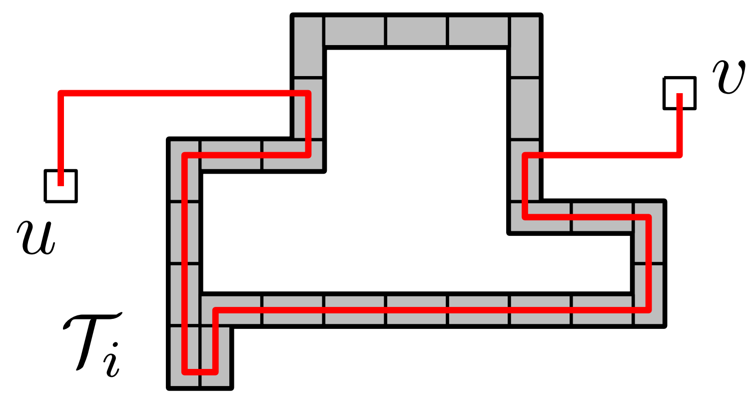

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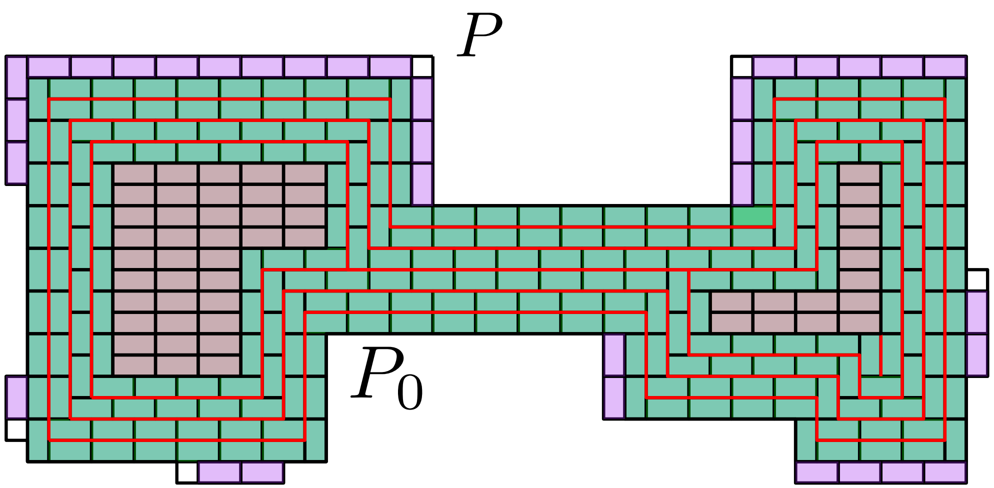

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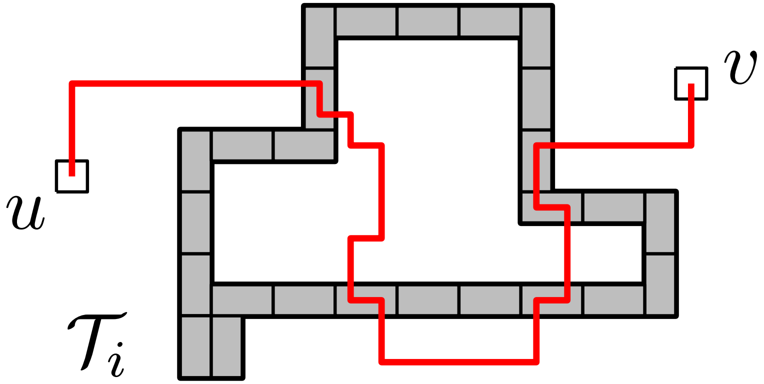
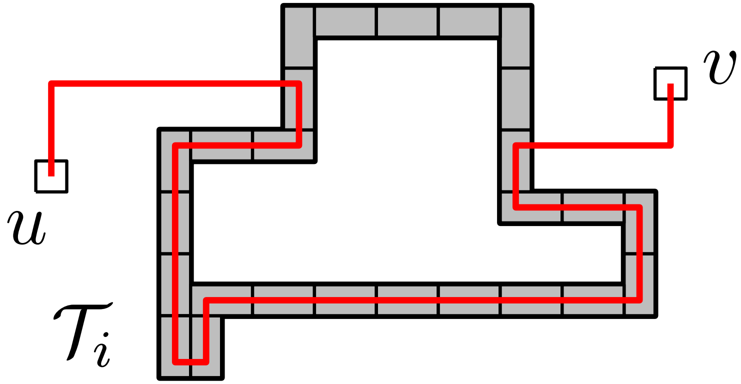
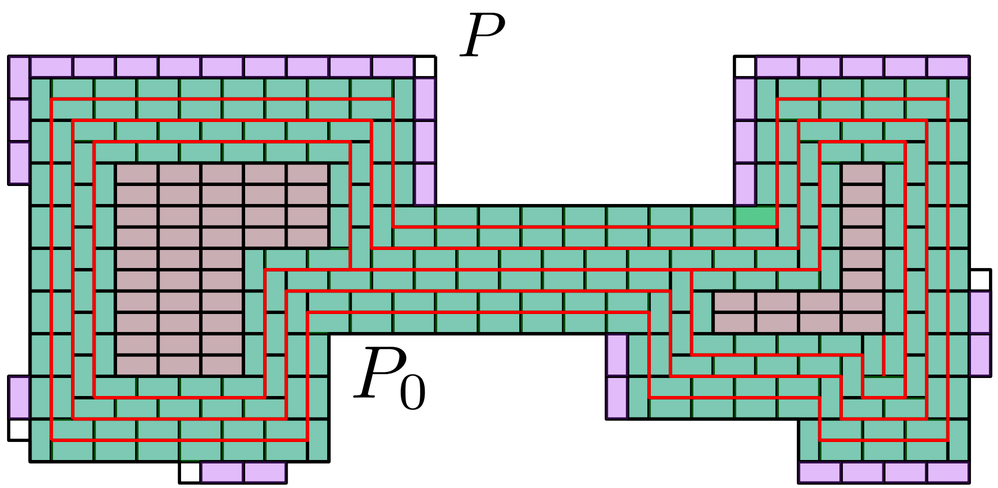




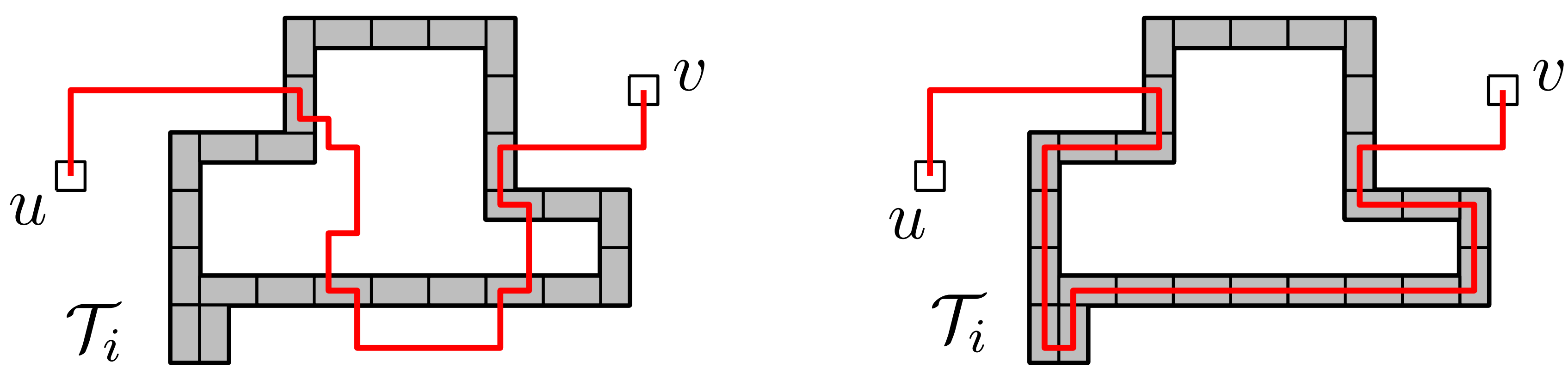
Has to repeat at most $r = \lfloor n/2 \rfloor$ times

Proof



Main Lemma. *There exists a maximum domino packing of P restricting to a tiling of Q.*

Important:
P has no holes $\Rightarrow$
u and v are both 'outside' each of the Hamiltonian cycles.



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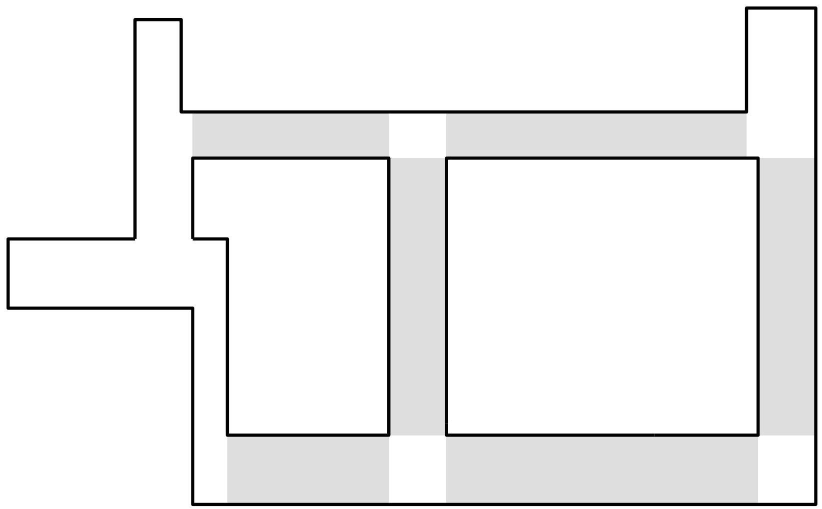
Reduced instance

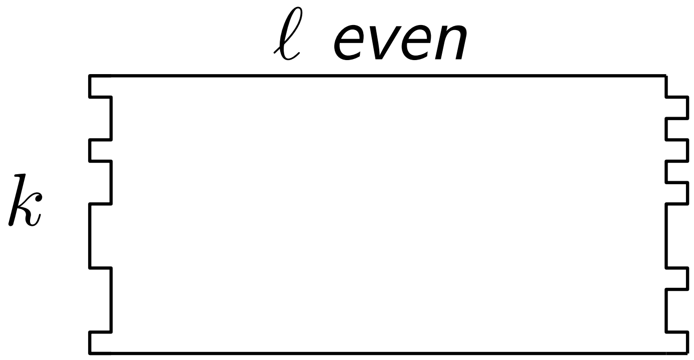
Issue: There can be exponentially long and narrow 'pipes' $\Rightarrow$ he size can be exponential.



However, any point of $P' = P \setminus Q$ is of distance $O(n)$ to $\partial P'$

Structural Result 2: Shortening Pipes

Consider 'pipes' of the form:



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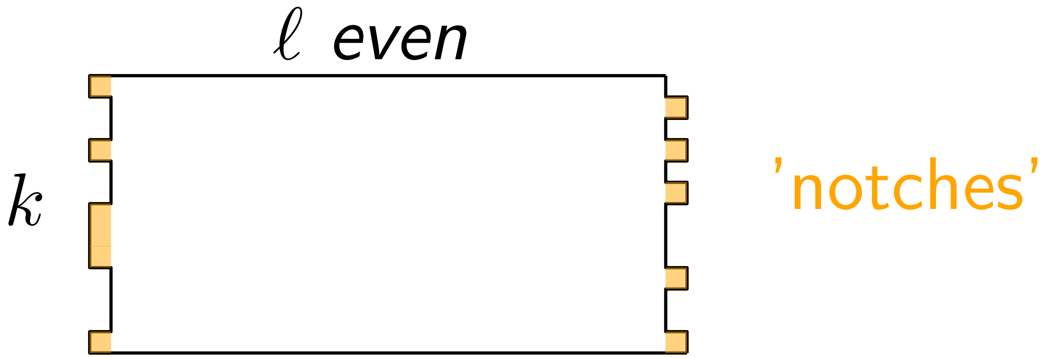

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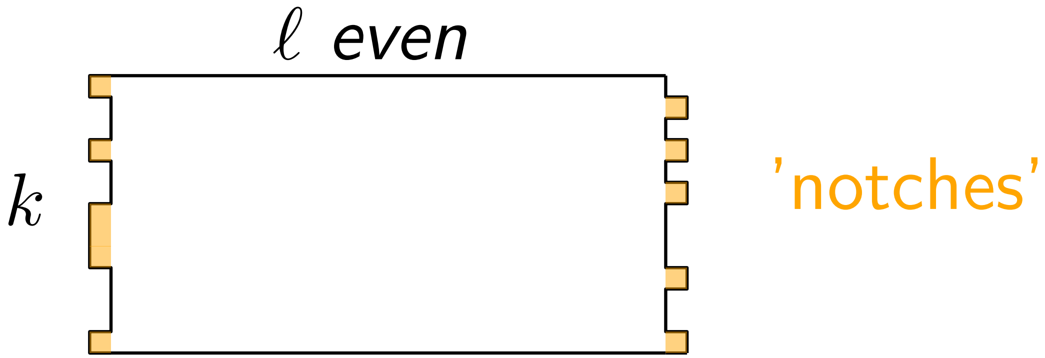

Color black and white in chessboard fashion with b black cells and w white cells. Assume $b \geq w$.

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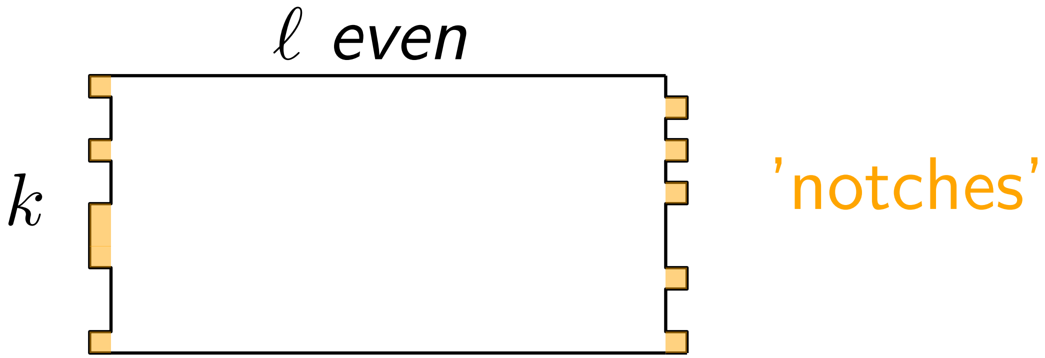

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Lemma. *If $\ell \geq 2k$, then the number of uncovered cells in a maximum domino packing is $b - w$*

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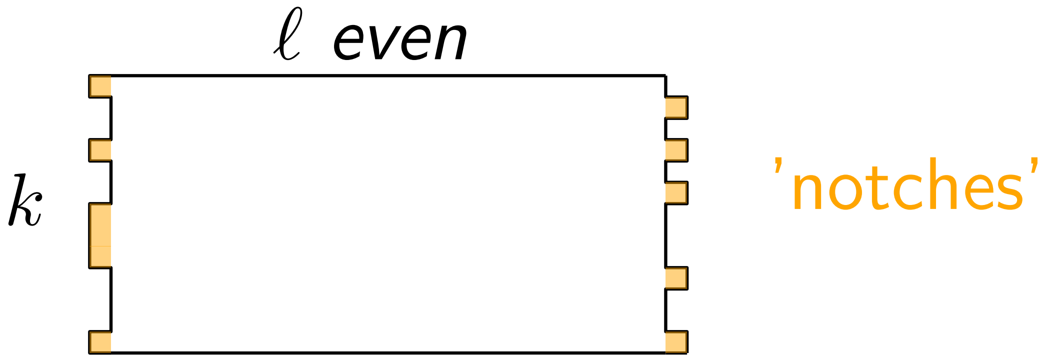

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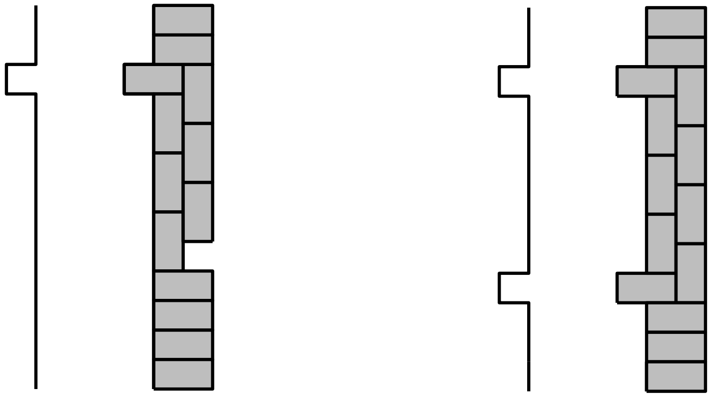

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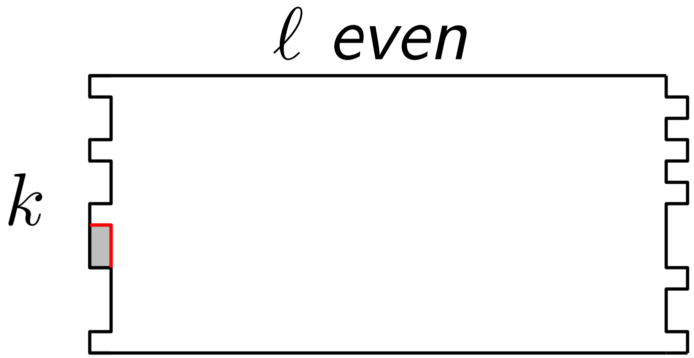

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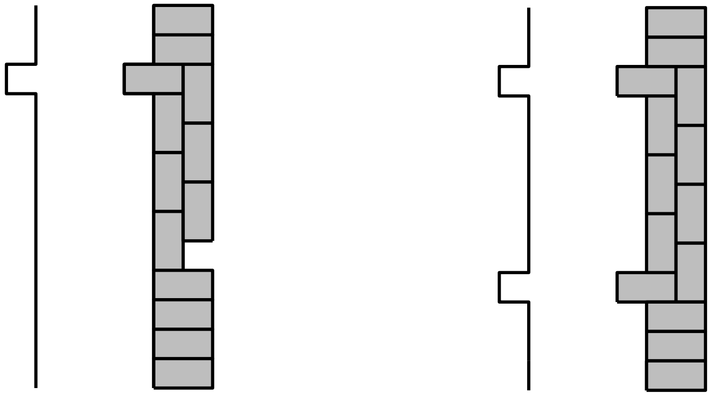

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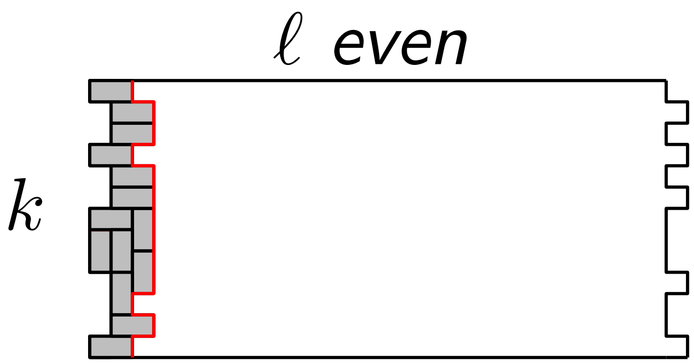

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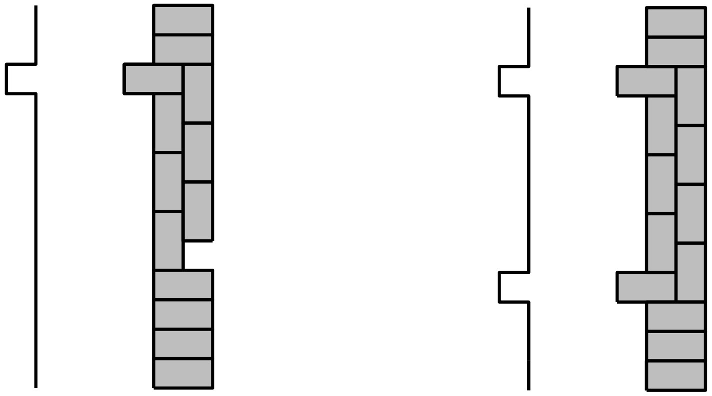

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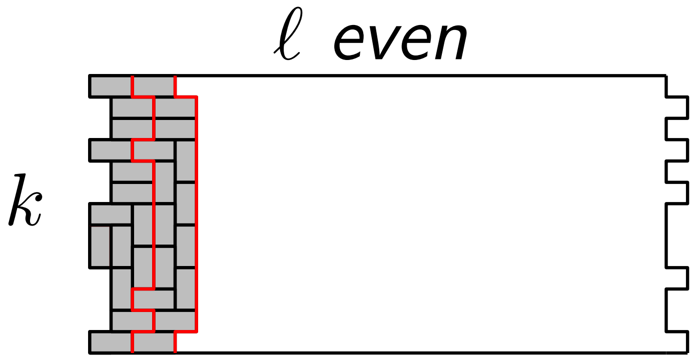

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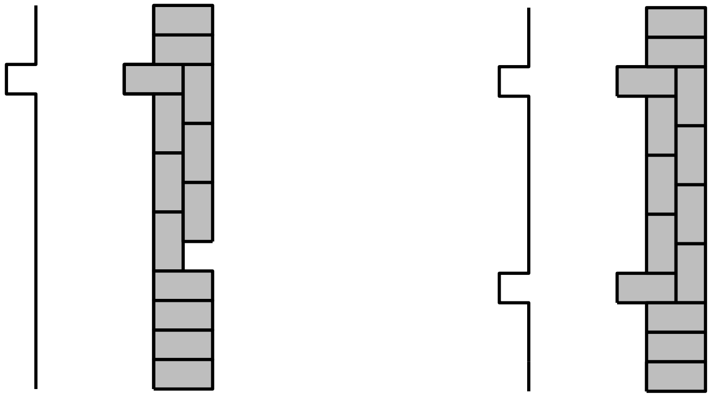

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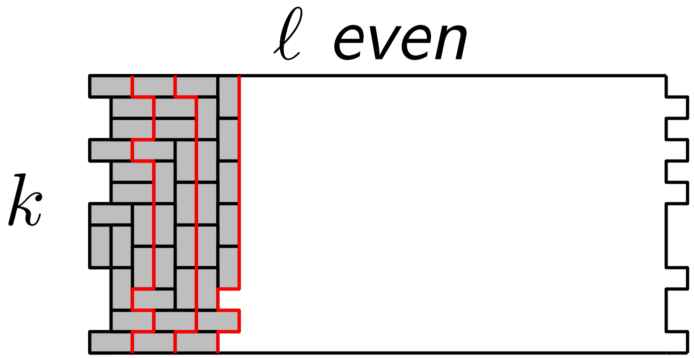

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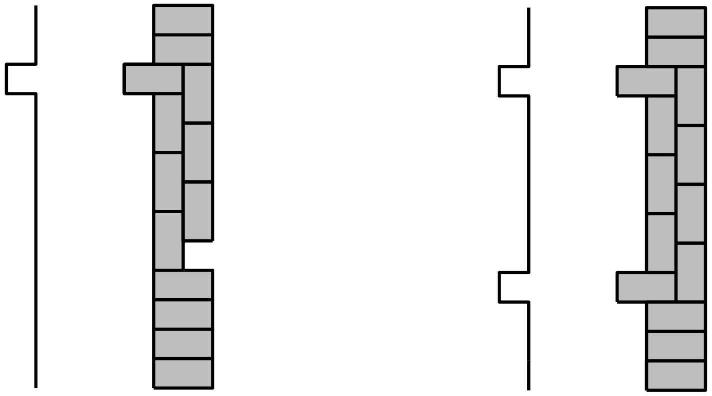

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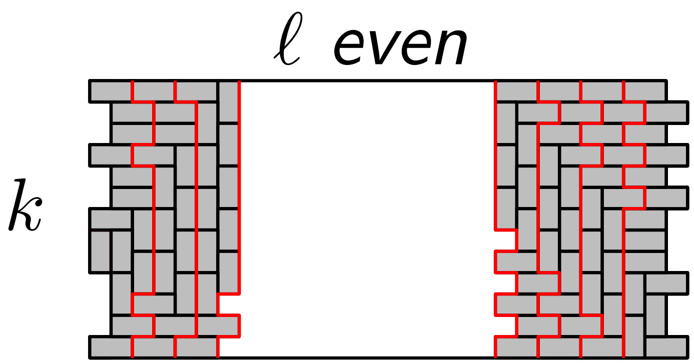

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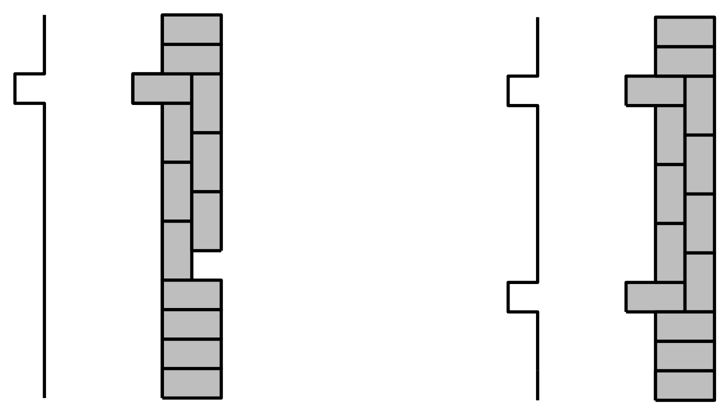

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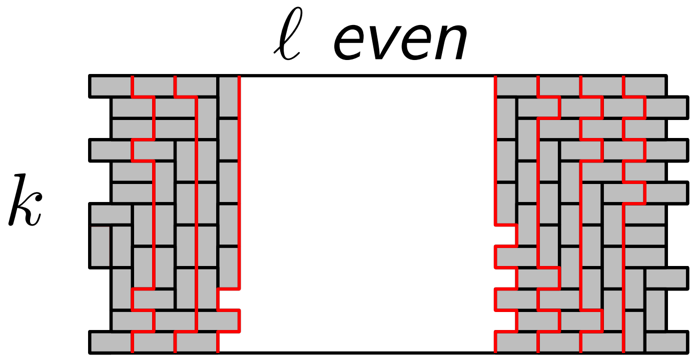

Now fill in horizontal dominos

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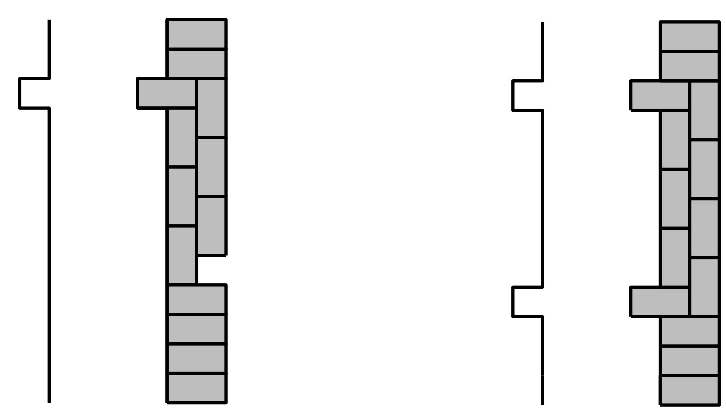

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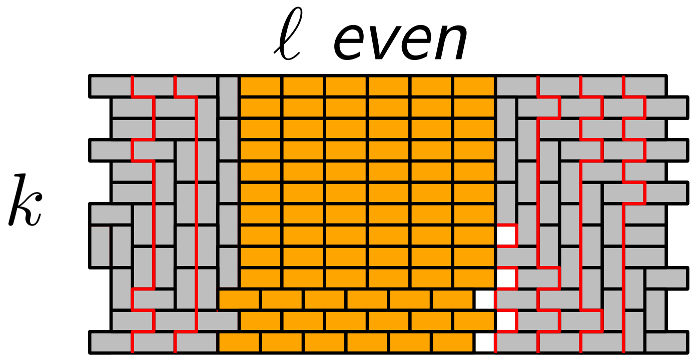

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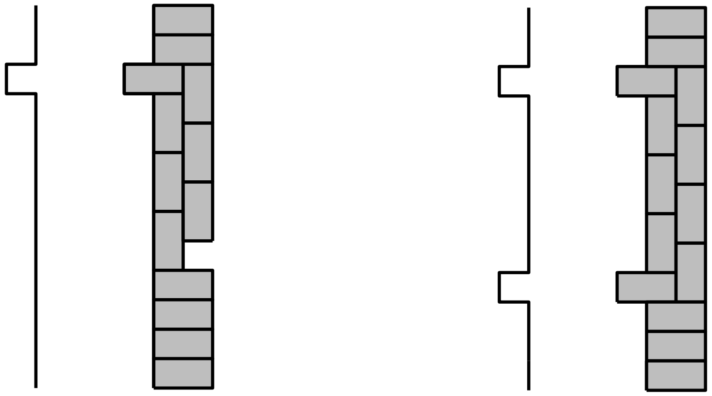

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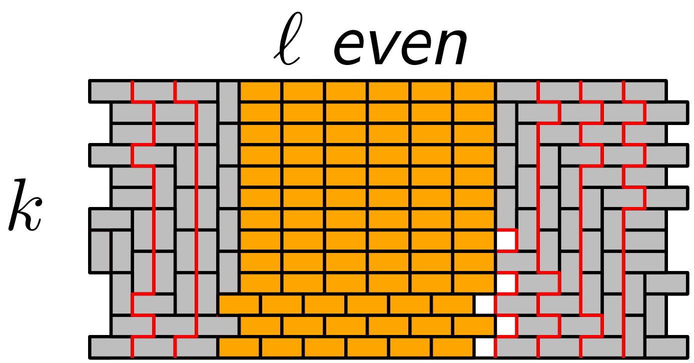

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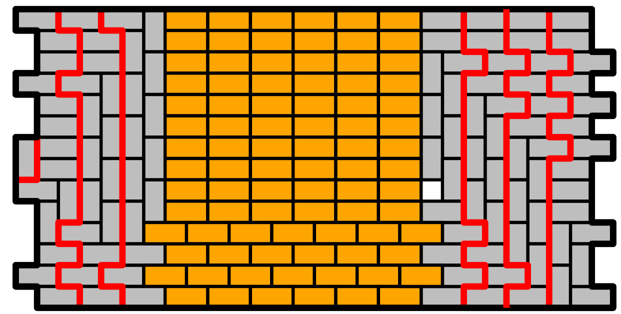

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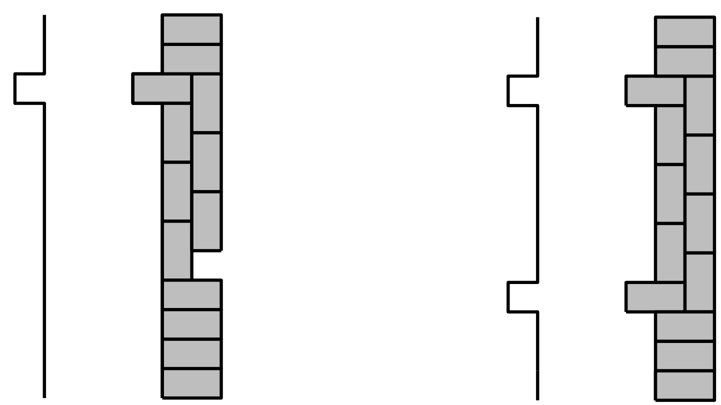
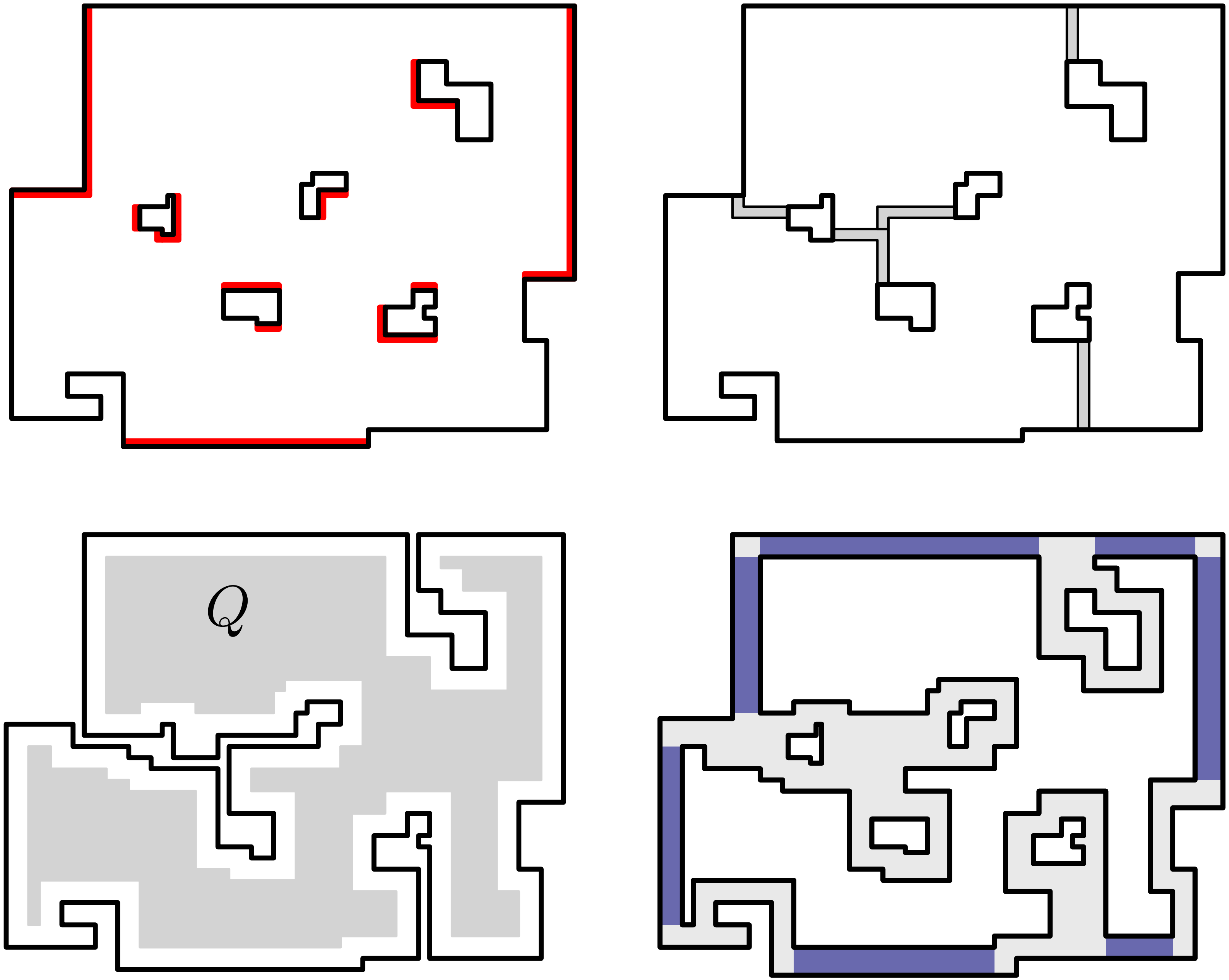


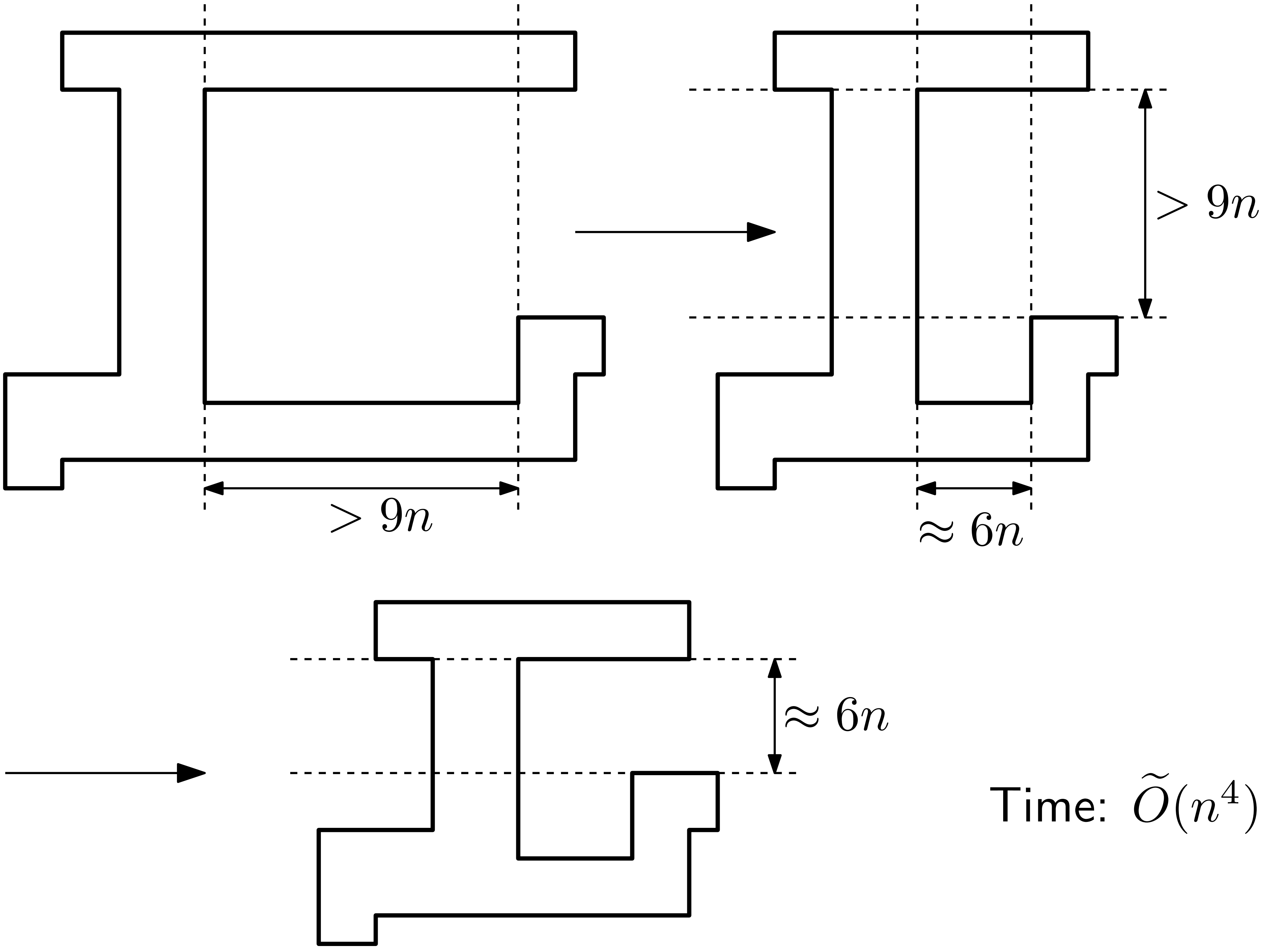
Running time

	Running time:
Compute P_0	$O(n \log n)$
Compute offset	$O(n \log n)$
Find long pipes	$O(n \log n)$
Find maximum matching	$O(n^3 \log^3 n)$

What if there are holes?

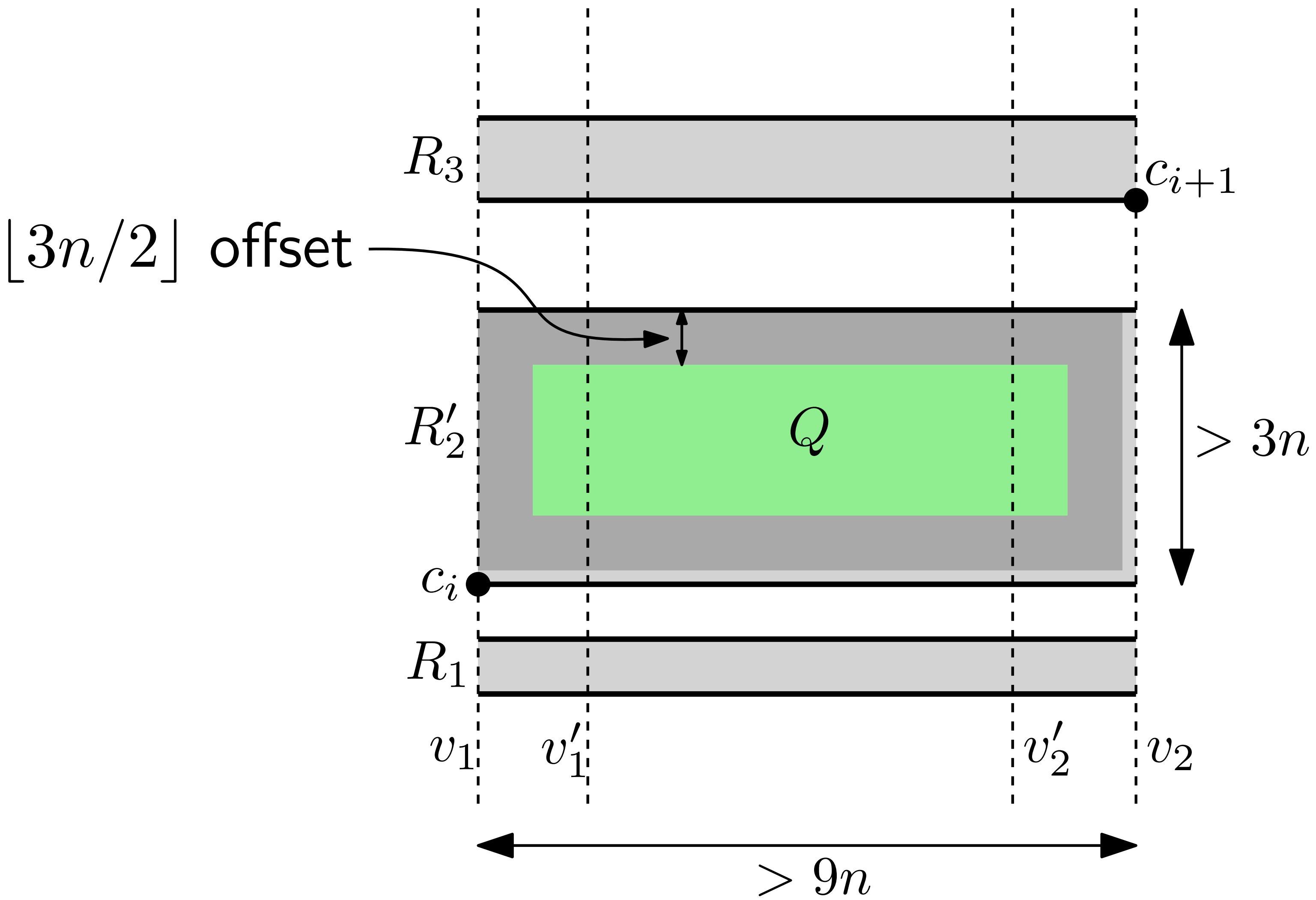


Simple algorithm



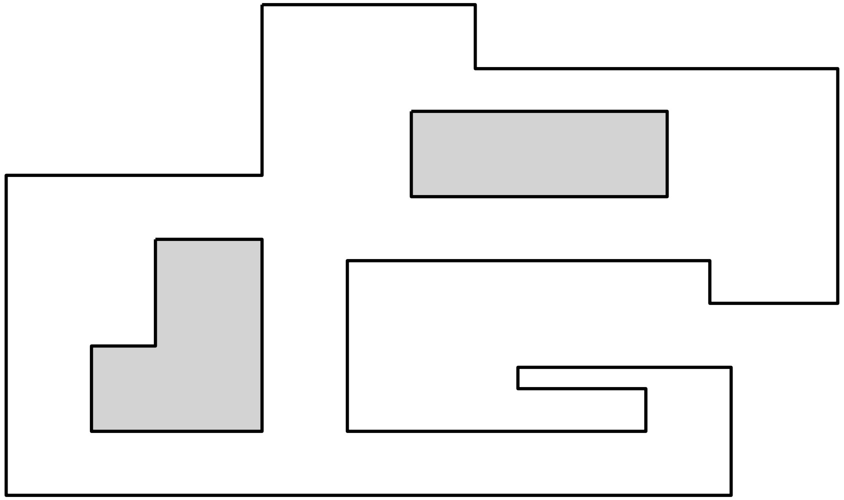
Time: $\widetilde{O}(n^4)$.

Correctness of simple algorithm



Open problems

Can domino tiling/packing be solved faster with a reduction to a flow problem?



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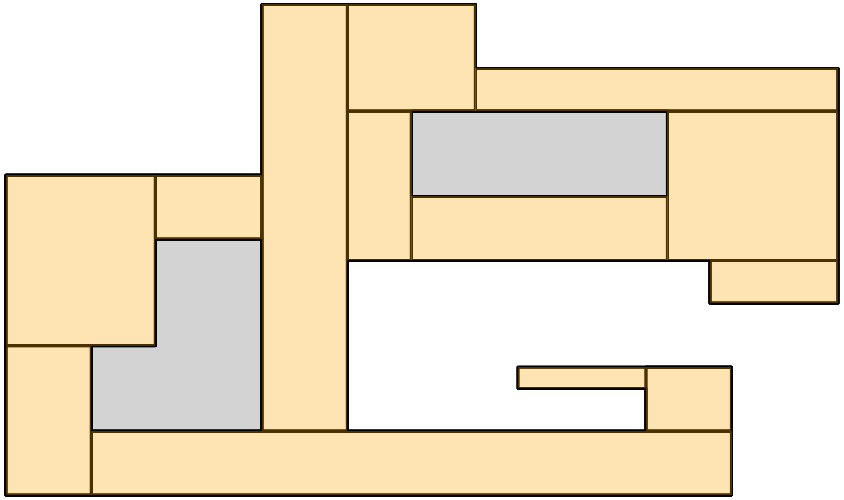

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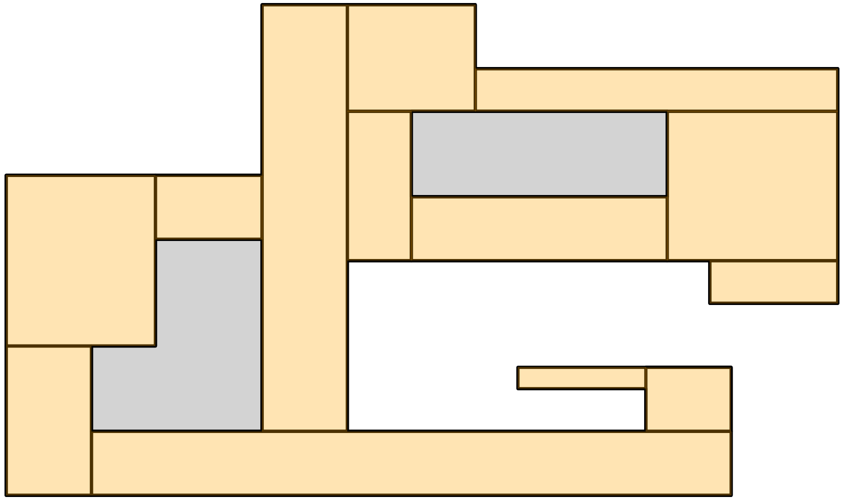

Packing 2×2 squares is NP-complete when P has holes. Can it be solved in polynomial time if P is hole-free?