Connections between Major Geometric Structures

Computational Geometry

Dan Halperin

Tel Aviv University

Overview

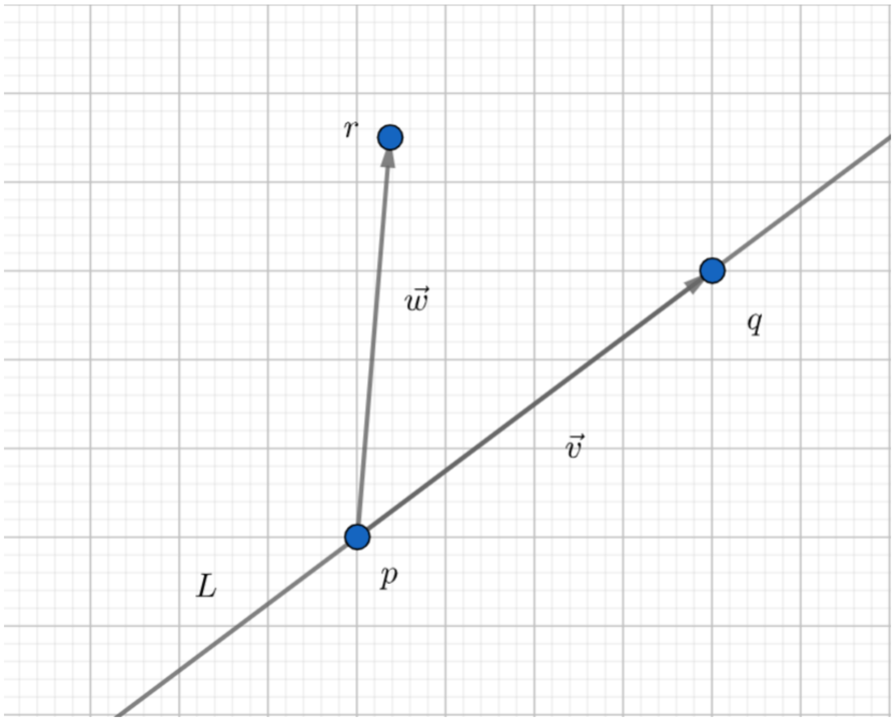
We begin by recalling several tools that we have studied throughout the course, learn a few more and then proceed with pointing out connections between the central structures that we have reviewed

Credits

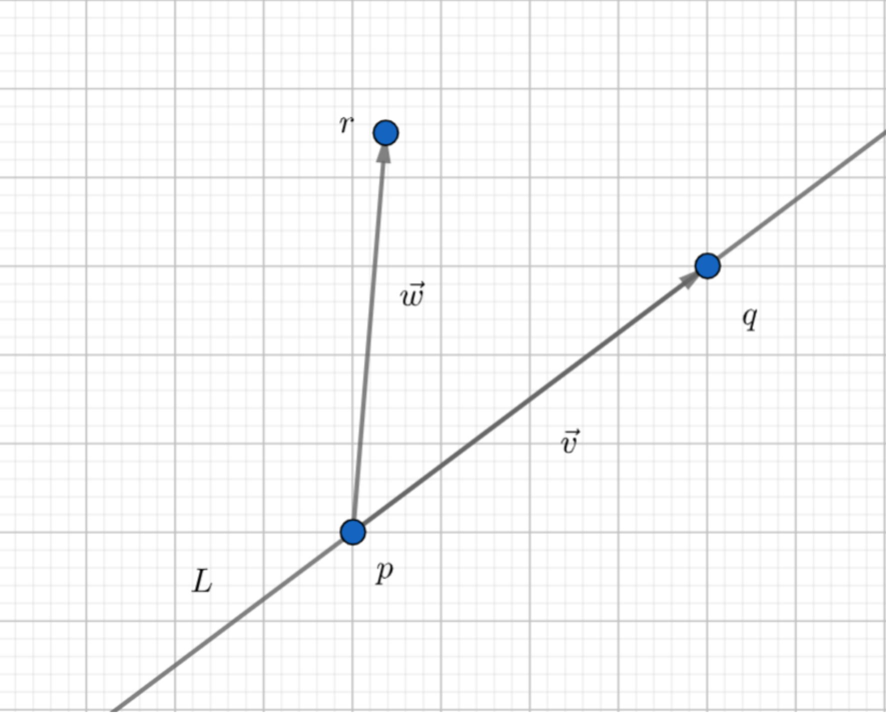
- some figures are taken from Computational Geometry Algorithms and Applications by de Berg et al [CGAA]
- the original figures are available at the book’s site: www.cs.uu.nl/geobook/

Orientation test

- given three points in the plane p, q, r, consider the line L through p and q oriented from p to q
- orientation (or side-of-line) test: is r to the left of L, on L, or to the right of L?



Orientation test, cont'd



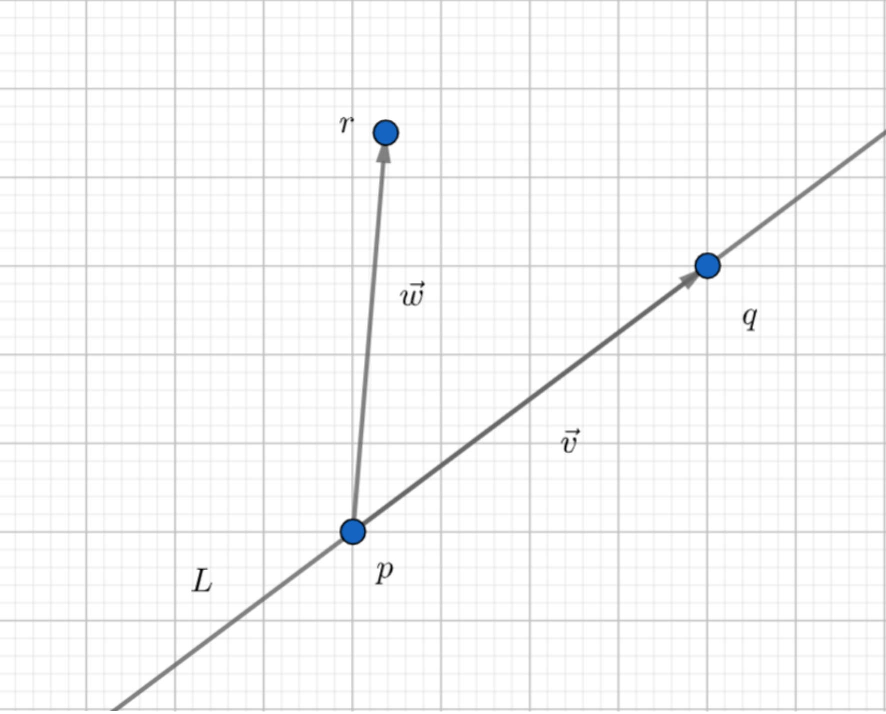
the vector product of $\vec{v}$ and $\vec{w}$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & 1 \\ w_x & w_y & 1 \end{vmatrix} = (v_x w_y - v_y w_x)\hat{k}$$

$$\vec{v} = q - p \Rightarrow v_x = q_x - p_x, \quad v_y = q_y - p_y$$
$$\vec{w} = r - p \Rightarrow w_x = r_x - p_x, \quad w_y = r_y - p_y$$

$$(v_x w_y - v_y w_x) = (q_x - p_x)(r_y - p_y) - (q_y - p_y)(r_x - p_x) \equiv \Delta(p, q, r)$$

Orientation test, cont'd



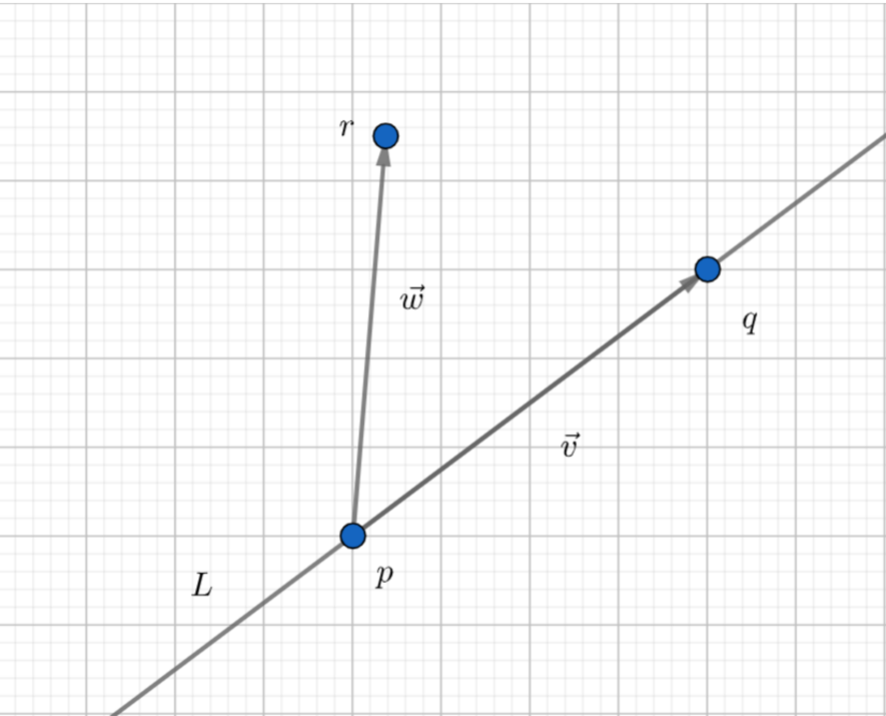
if $\Delta(p, q, r) > 0$ then r is to the **left** of $L(p, q)$

if $\Delta(p, q, r) = 0$ then r is **on** of $L(p, q)$

if $\Delta(p, q, r) < 0$ then r is to the **right** of $L(p, q)$

Orientation test, equivalent formulation

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & 1 \\ w_x & w_y & 1 \end{vmatrix} = \begin{vmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{vmatrix}$$



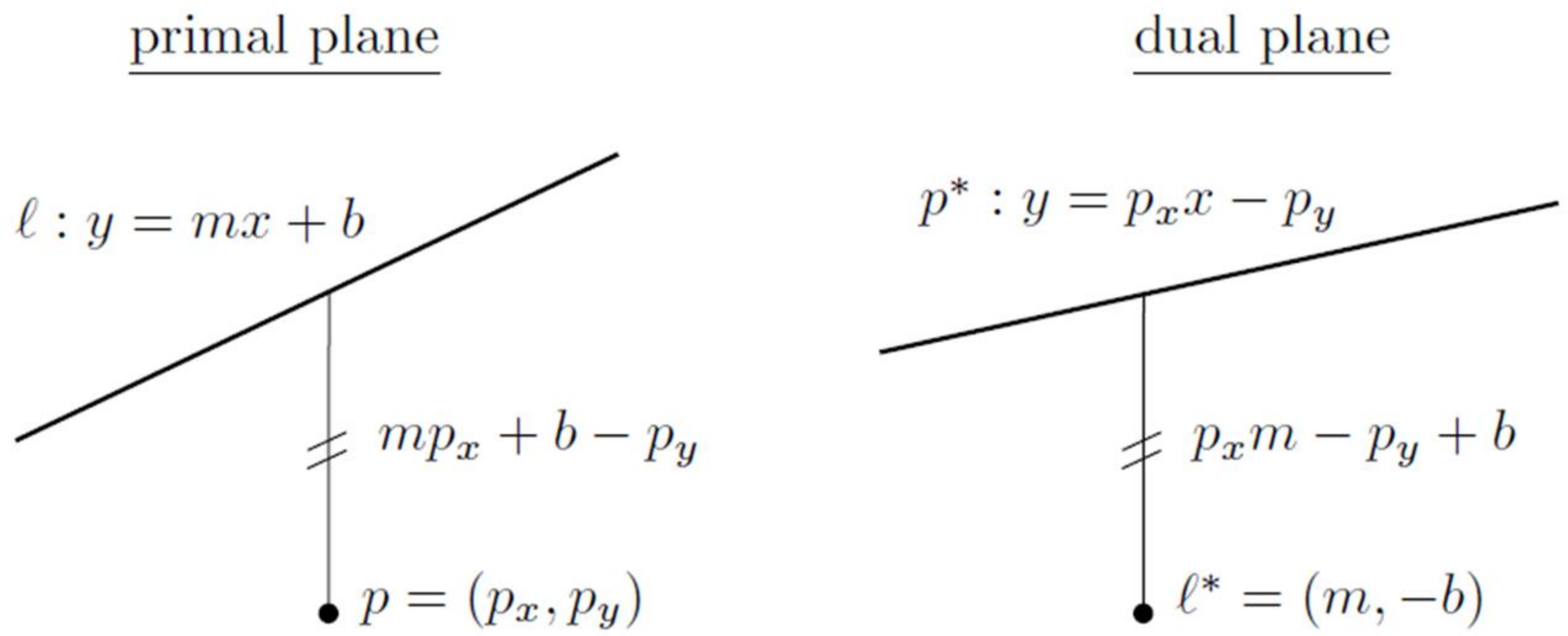
Orientation test in higher dimensions

- in 3D: on which side of the *oriented plane* $H(p,q,r)$ does the point s lie?

$$\begin{vmatrix} p_x & p_y & p_z & 1 \\ q_x & q_y & q_z & 1 \\ r_x & r_y & r_z & 1 \\ s_x & s_y & s_z & 1 \end{vmatrix} \quad >, \quad <, \quad = \quad 0?$$

- in R^d: on which side of an oriented hyperplane containing d points does another point lie? the determinant of a $d+1 \times d+1$ matrix

Point-line duality in the plane



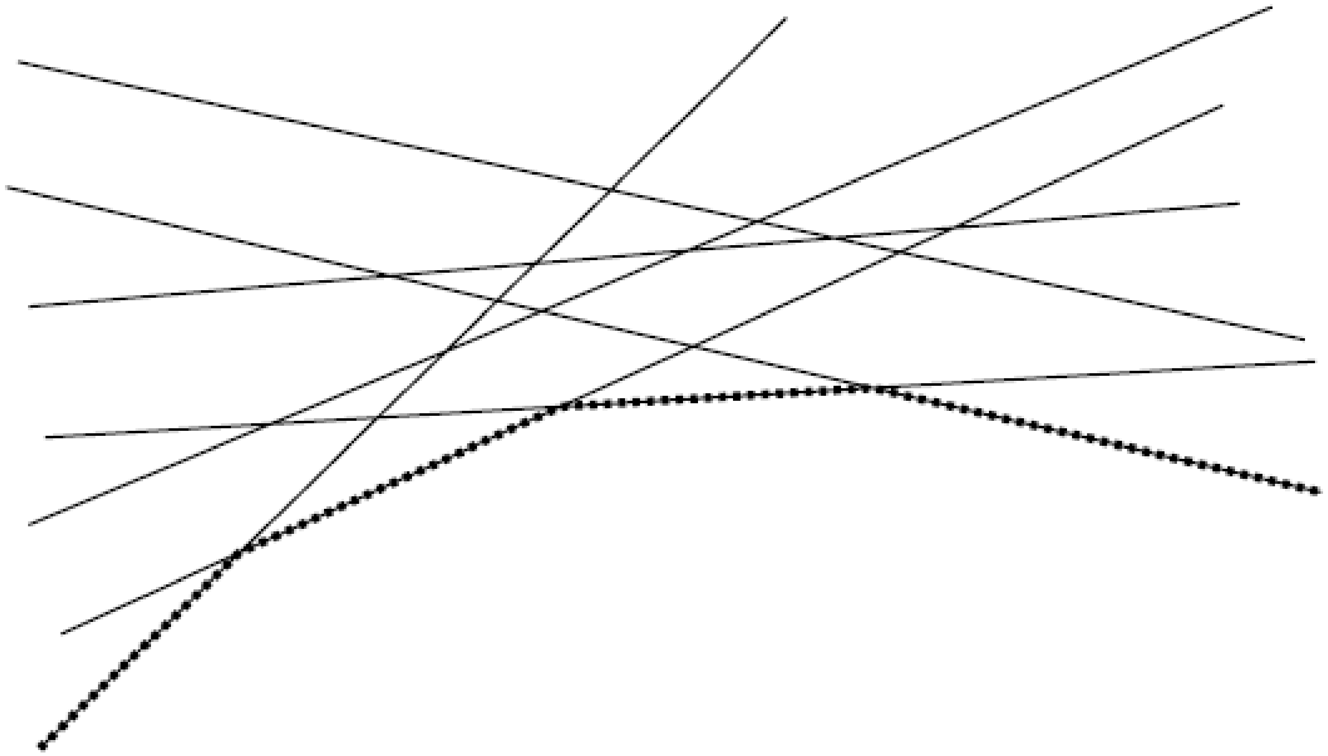
Duality preserves vertical distances

Duality in higher dimensions

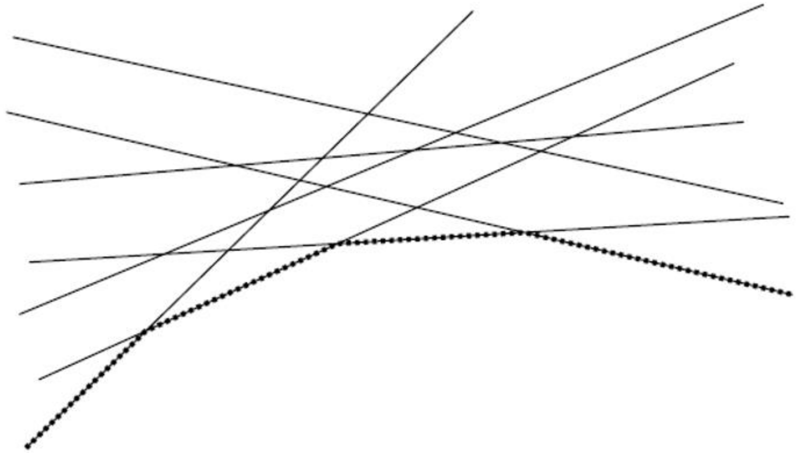
- in R^d, duality between
  - the point $(a_1, a_2, \dots, a_d)$ and
    the hyperplane $x_d = a_1 x_1 + a_2 x_2 + \cdots + a_{d-1} x_{d-1} - a_d$
  - the hyperplane $x_d = a_1 x_1 + a_2 x_2 + \cdots + a_{d-1} x_{d-1} + a_d$ and
    the point $(a_1, a_2, \dots, -a_d)$

preserves above/below/incidence relations, vertical distance

Arrangements of lines and their lower envelope

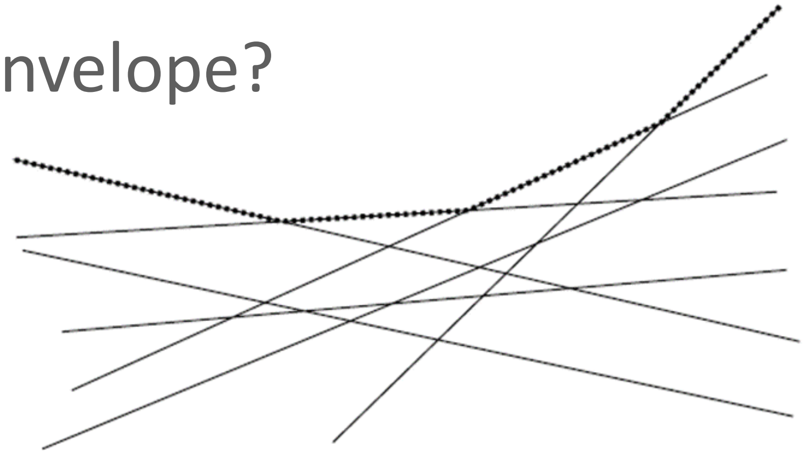


Envelopes



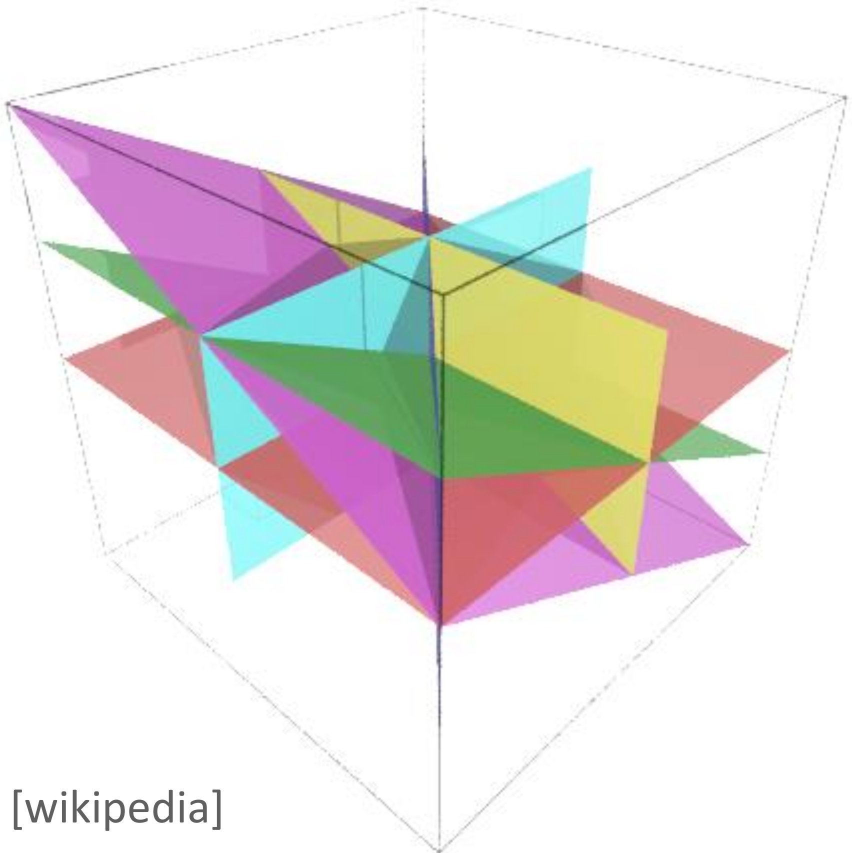
- arrg of n lines
- what is the shape below the lower envelope?
- what is the exact maximum complexity of the envelope?

- what is the shape above the upper envelope?
- what is the exact maximum complexity of the envelope?



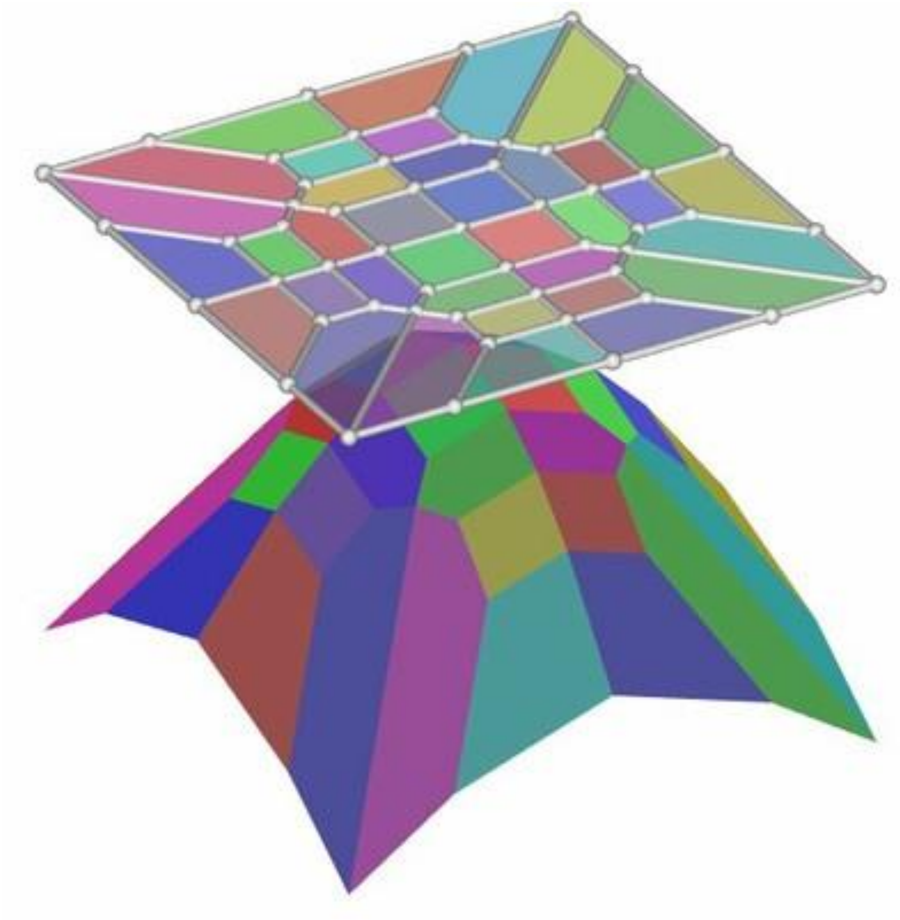
Arrangements of planes and their lower envelope

- arrg of n planes, H
- how does the arrg look like on one plane in H?
- how complex is one such arrg?
- how complex is the arrg of planes
- how many 3D cells it has?
- the upper and lower envelope: shape and complexity



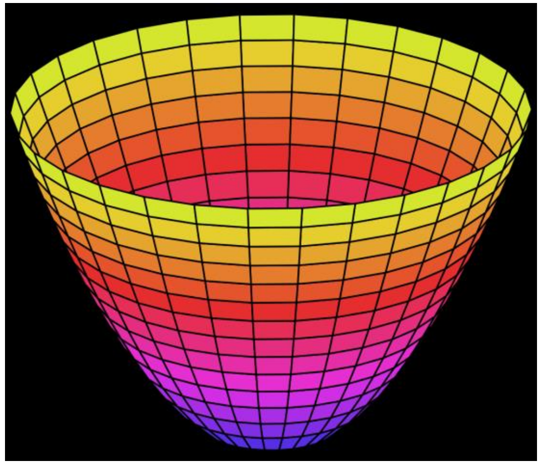
[wikipedia]

Degenerate lower envelope of planes and its minimization diagram



- we assume henceforth general position

The lifting transform



[wikipedia]

- the lifting transform maps points in R^d to objects (points or hyperplanes) in R^{d+1}
- we will focus on the plane, and the vertical projection of planar points onto the *unit paraboloid* U in R^3:

  $U: z = x^2 + y^2$
- vertical cross-sections of U are parabolas, horizontal cross-sections are circles
- $LT: p(x, y) \longmapsto \hat{p}(x, y, x^2 + y^2)$

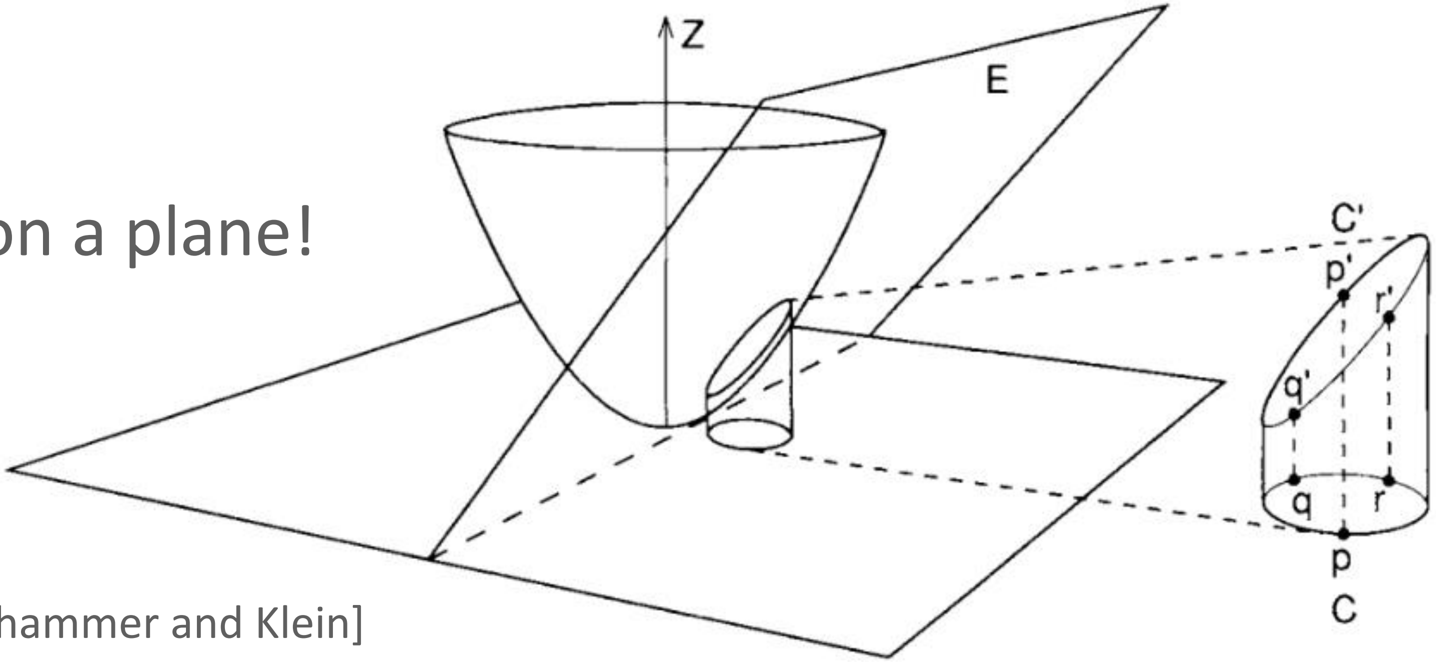
Lifting a circle

- $LT: p(x, y) \longmapsto \hat{p}(x, y, x^2 + y^2)$
- $C(a, b, r)$ is a circle in the plane with center at (a, b) and radius r
- $LT: C(a, b, r) \longmapsto ?$
- C: $(x - a)^2 + (y - b)^2 = r^2$
- C: $x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$
- $\hat{C}$ is on U, therefore in $\hat{C}$ we can replace $x^2 + y^2$ by z, to obtain
- $z = 2ax + 2by - (a^2 + b^2 - r^2)$

Lifting a circle, cont'd

- $z = 2ax + 2by - (a^2 + b^2 - r^2)$

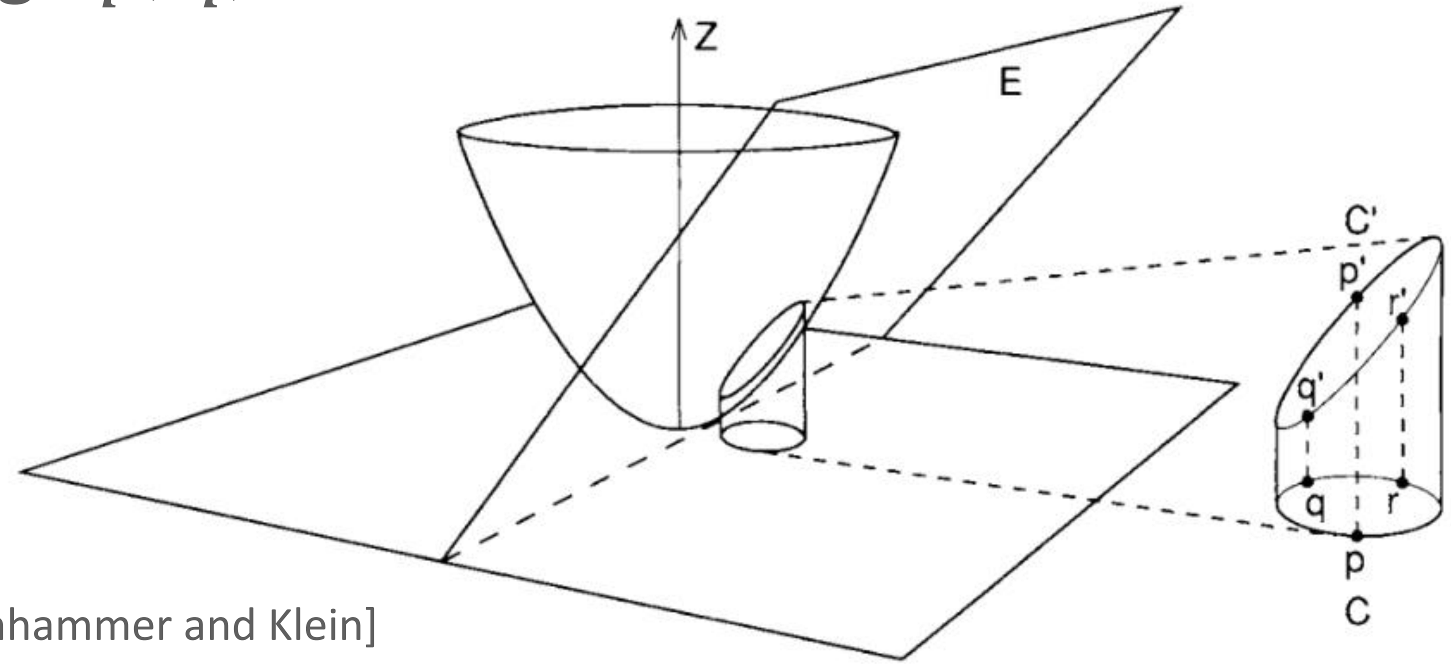
- the lifted circle $\widehat{C}$ resides on a plane!



[Aurenhammer and Klein]

Corollary

- Let p, q, r, s be points in the plane. The point s lies inside the circle though p, q, r iff the point $\hat{s}$ lies below the plane through $\hat{p}, \hat{q}, \hat{r}$.



[Aurenhammer and Klein]

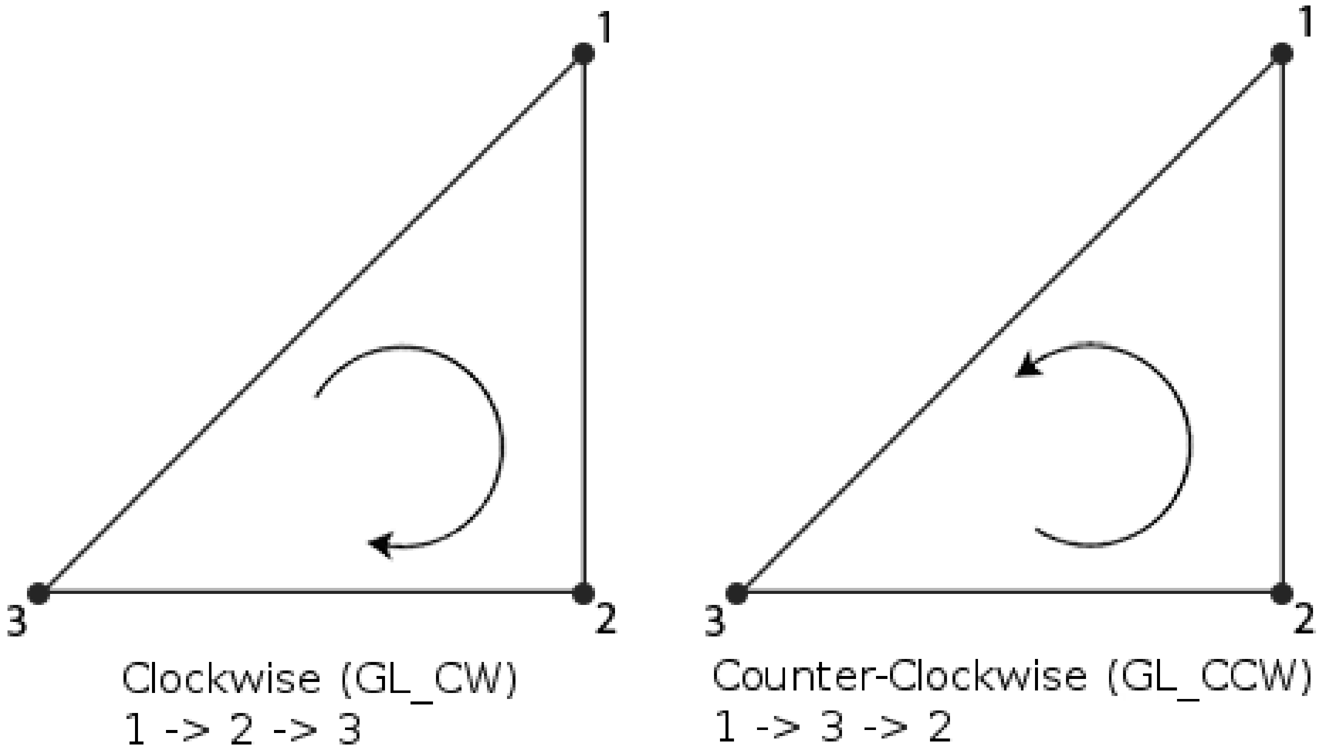
Point-in-circle test

- without computing the center or radius of the circle
- recall, for p, q, r, s points in R^3:

$$\begin{vmatrix} p_x & p_y & p_z & 1 \\ q_x & q_y & q_z & 1 \\ r_x & r_y & r_z & 1 \\ s_x & s_y & s_z & 1 \end{vmatrix} \quad >, \quad <, \quad = \quad 0?$$

  determines on which side of the plane $H(p, q, r)$ through p, q, r does s lie
- we still need to orient the plane $H(p, q, r)$

Orienting triangles



[wikipedia]

How exactly?

$$\Phi(p,q,r,s) = \begin{vmatrix} p_x & p_y & p_z & 1 \\ q_x & q_y & q_z & 1 \\ r_x & r_y & r_z & 1 \\ s_x & s_y & s_z & 1 \end{vmatrix}$$

if $\Phi(p,q,r,s) > 0$ then s is on the side of $H(p,q,r)$ from which (p,q,r) is oriented **counterclockwise**

if $\Phi(p,q,r,s) = 0$ then s is **on** $H(p,q,r)$

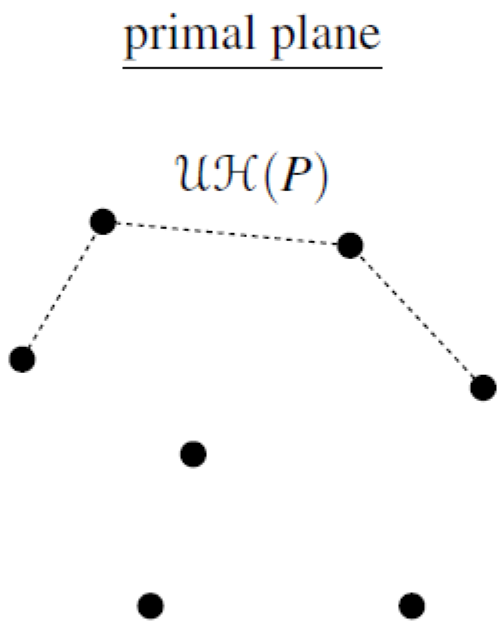
if $\Phi(p,q,r,s) < 0$ then s is on the side of $H(p,q,r)$ from which (p,q,r) is oriented **clockwise**

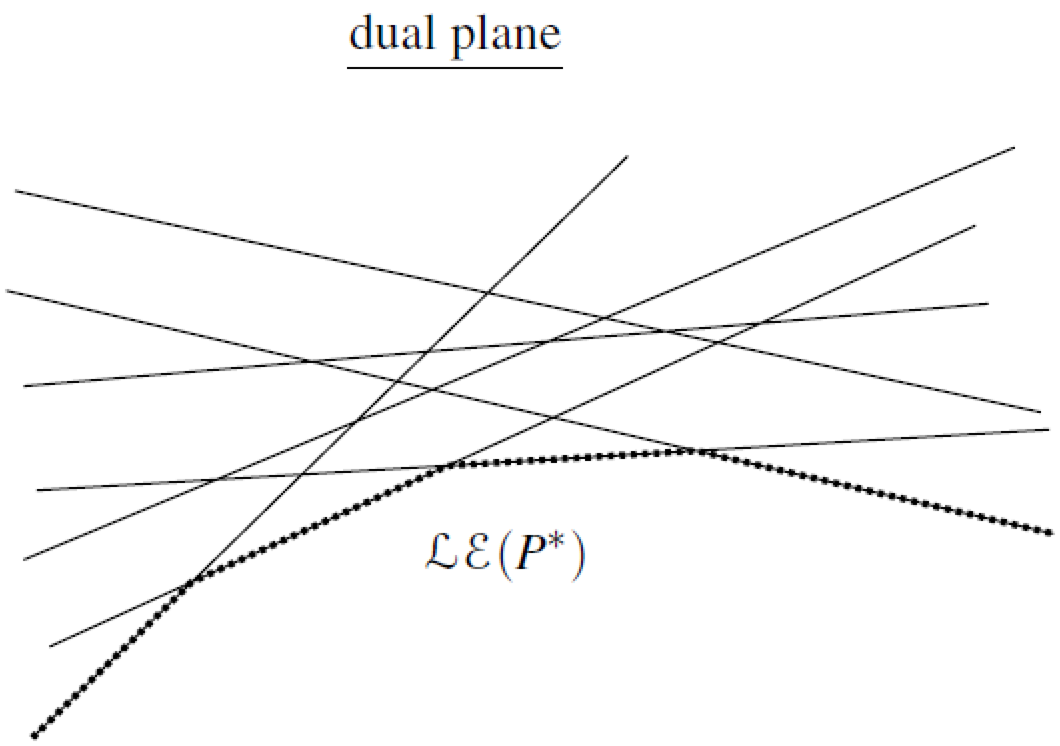
Point-in-circle test

- recall: For p, q, r, s points **in the plane**, the point s lies inside the circle though p, q, r iff the point $\hat{s}$ lies below the plane through $\hat{p}, \hat{q}, \hat{r}$
- assume that (p, q, r) are oriented **clockwise**
- then the point s is inside the circle the circle through p, q, r in the plane iff $\Phi(\hat{p}, \hat{q}, \hat{r}, \hat{s}) > 0$, namely

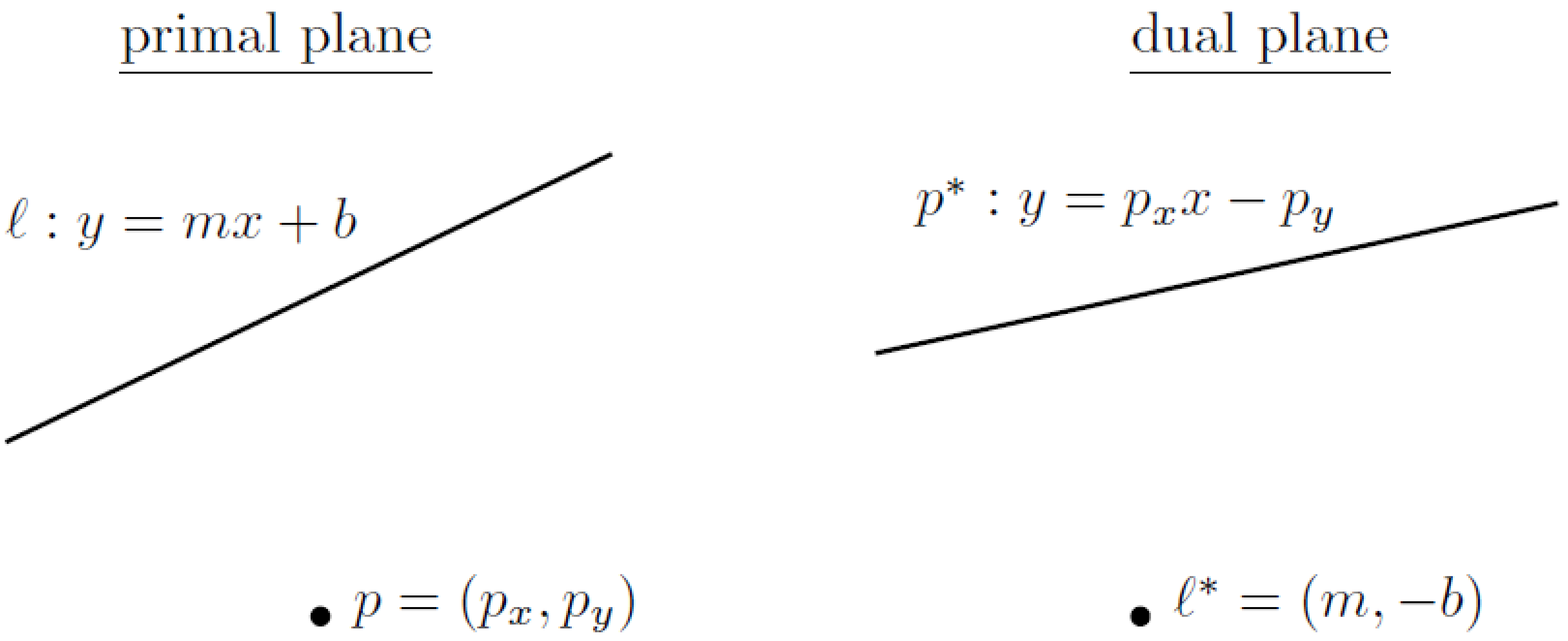
$$\Phi(\hat{p}, \hat{q}, \hat{r}, \hat{s}) = \begin{vmatrix} p_x & p_y & p_x^2 + p_y^2 & 1 \\ q_x & q_y & q_x^2 + q_y^2 & 1 \\ r_x & r_y & r_x^2 + r_y^2 & 1 \\ s_x & s_y & s_x^2 + s_y^2 & 1 \end{vmatrix} > 0$$

Connection: hulls and envelopes





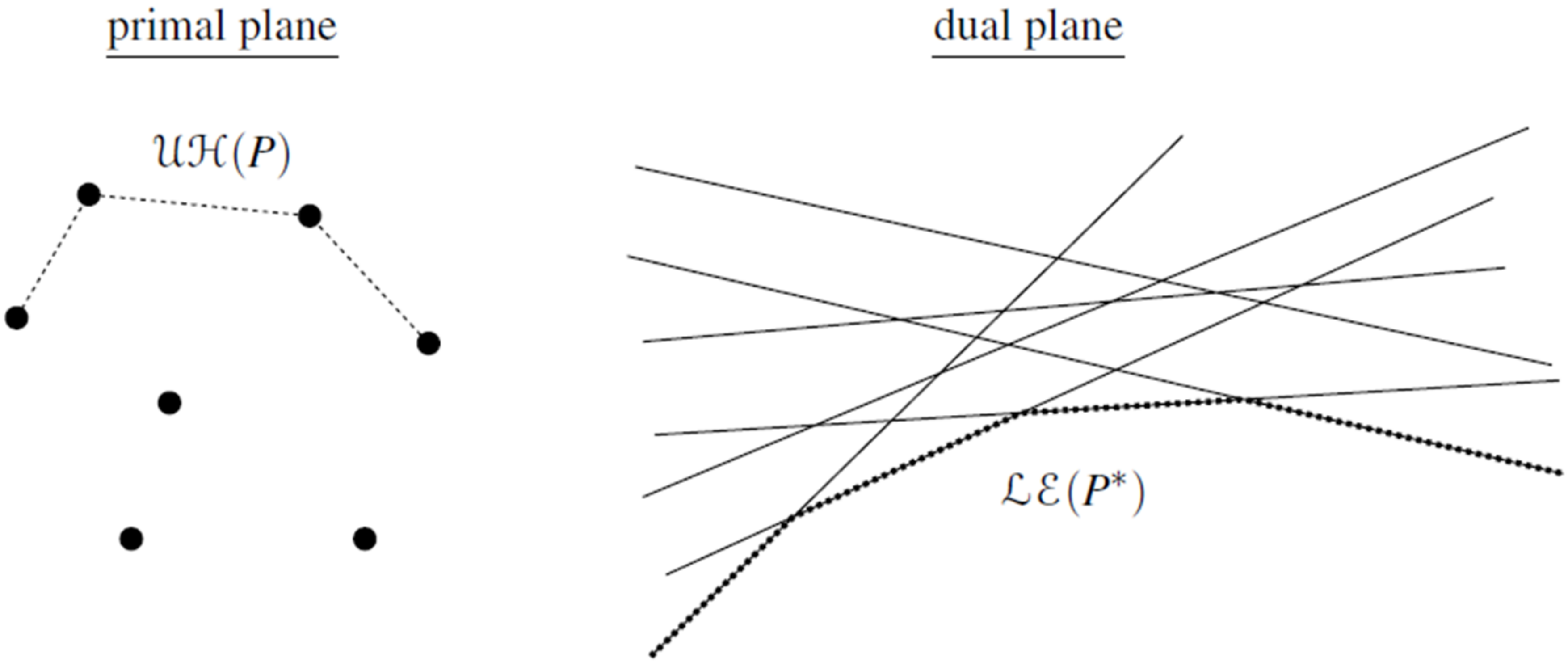
Recall



point $p = (p_x, p_y) \mapsto$ line $p^* : y = p_x x - p_y$

line $\ell : y = mx + b \mapsto$ point $\ell^* = (m, -b)$

Therefore: the upper hull corresponds to the lower envelope



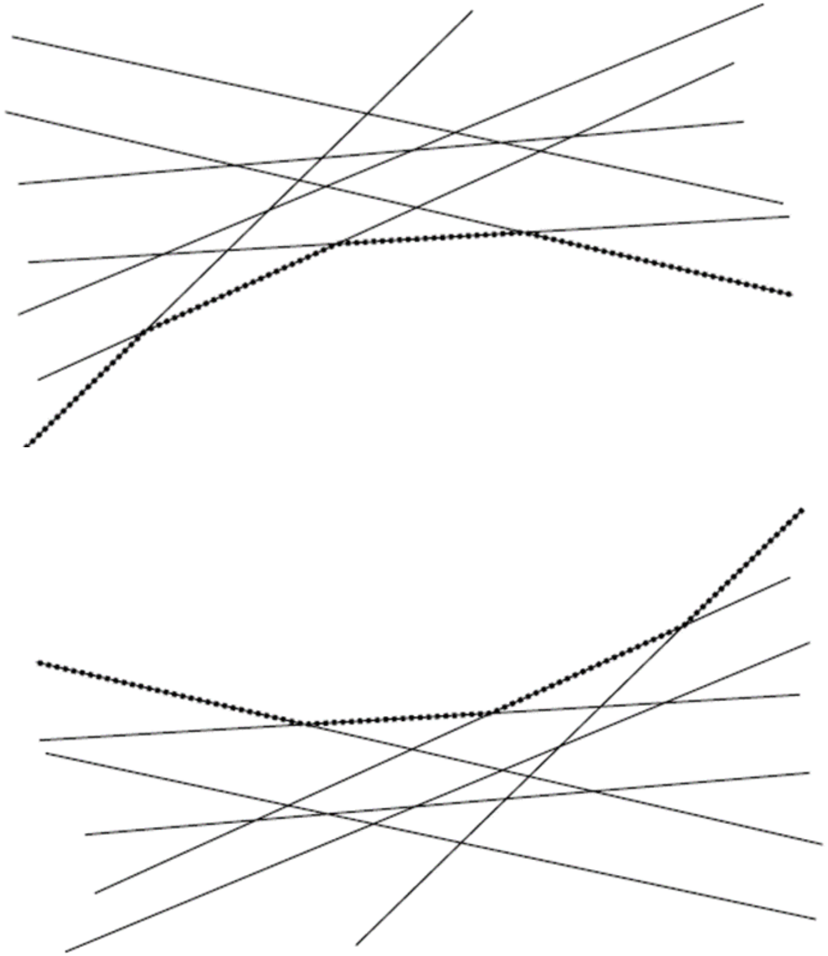
- hull edges correspond to envelope breakpoints
- in what order?

Hulls and envelopes

- under "our" duality the upper hull of points in P corresponds to the lower envelope of the dual lines P^* and the lower hull correspond to the upper envelope
- holds in any dimension
- in R^3 for a set P of points:
  - a vertex of the upper hull of the points in P (which is a point of P) corresponds to a face of the lower envelope of the planes in P^*
  - a facet of the upper hull corresponds to a vertex of the lower envelope
  - an edge of the upper hull corresponds to an edge of the lower envelope: the edge connecting two vertices v_1, v_2 of the hull corresponds to the joint edge on the boundary of the faces of the lower envelope that correspond to v_1, v_2

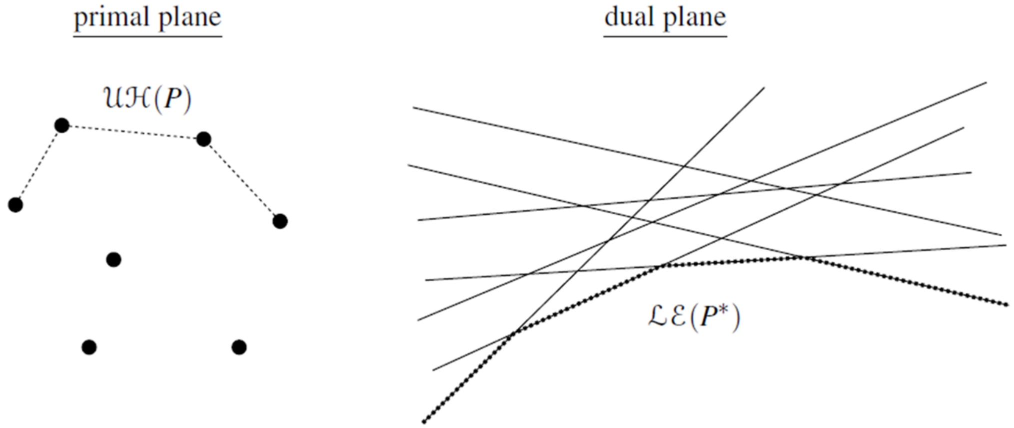
Convex hull vs. intersection of half-planes

- recall: the region below the lower envelope (or above the upper envelope) of lines is the intersection of half-planes
- question: can we use a convex-hull algorithm to compute the intersection of half-planes (tricky)?



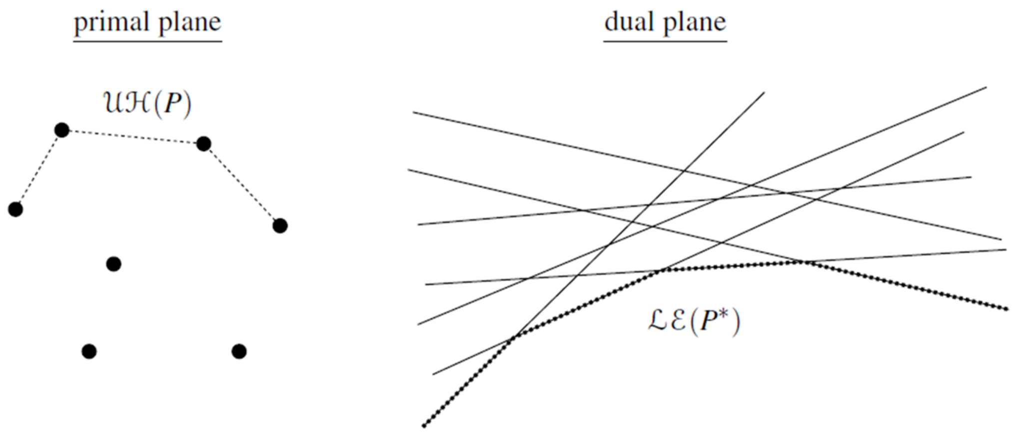
Convex hull vs. intersection of half-planes, cont'd

- Q: can we use a convex-hull algorithm to compute the intersection of half-planes?
- A: yes, but with care: we need to separate the half-planes into (i) upward facing, (ii) downward facing, and (iii) bounded by vertical lines
- for (i) and (ii) we can dualize the bounding lines and compute the relevant hull
- for (iii) ?



Convex hull vs. intersection of half-planes, cont'd

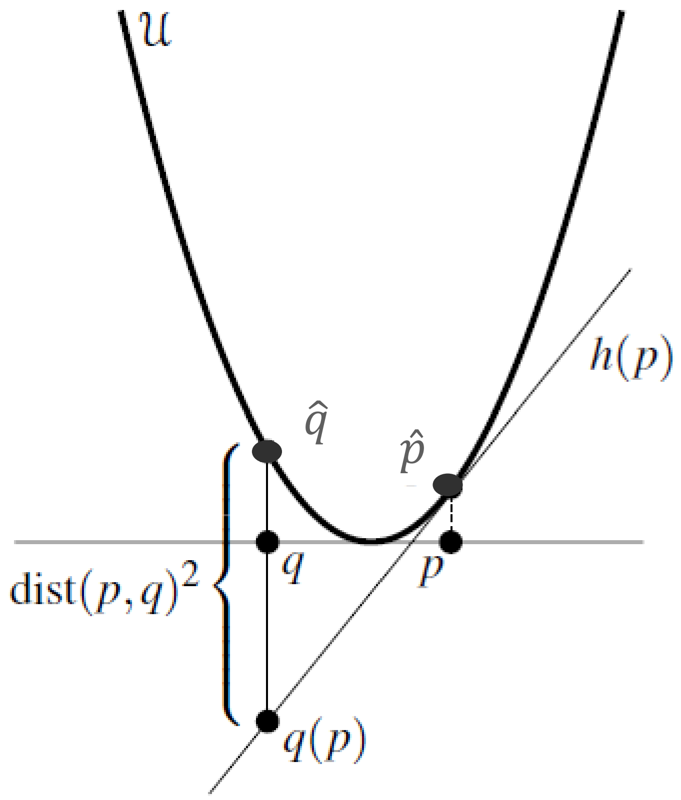
- corollary: computing the intersection of n half-planes in the plane requires $\Omega(n \log n)$ time
- notice: the convex hull is never empty while the intersection of half-planes can be
- holds in any dimension



Connection: Voronoi diagrams and upper envelopes in one dimension higher

- U is the unit paraboloid in R^3
- we lift the planar point $p(p_x, p_y)$ to $\hat{p}$ on U
- consider the following plane $h(p)$ that contains the point $\hat{p}(p_x, p_y, {p_x}^2 + {p_y}^2)$:

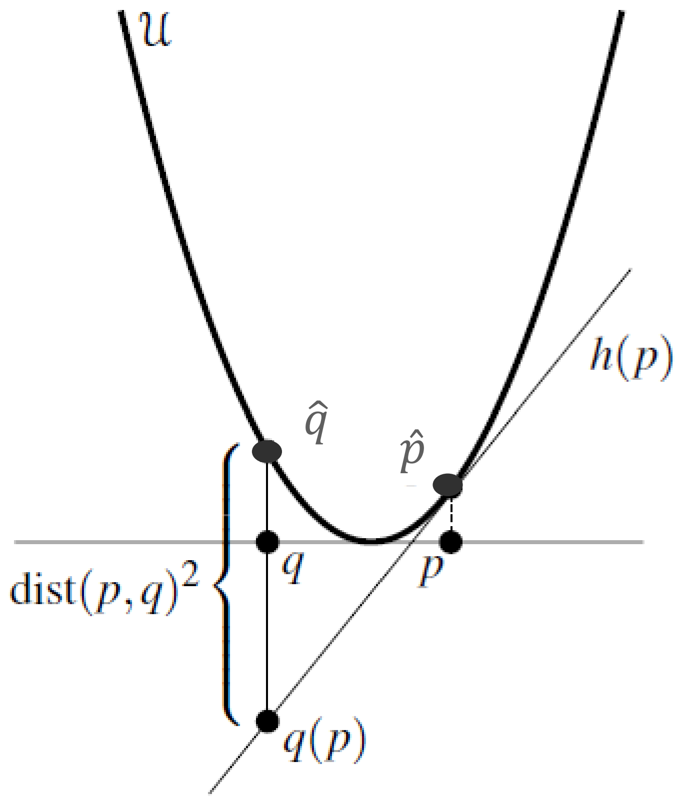
  $h(p): z = 2p_x\mathrm{x} + 2p_y y - ({p_x}^2 + {p_y}^2)$
- lift another point q in the plane to $\hat{q}$
- let $q(p)$ be the point where the vertical line through q intersect $h(p)$



The (vertical) distance between $\hat{q}$ and $q(p)$

- $h(p): z = 2p_x \mathrm{x} + 2p_y y - ({p_x}^2 + {p_y}^2)$
- $\hat{q}(q_x, q_y, {q_x}^2 + {q_y}^2)$
- $\Delta z = {q_x}^2 + {q_y}^2 - 2p_x q_x - 2p_y q_y + ({p_x}^2 + {p_y}^2) = (q_x - p_x)^2 + (q_y - p_y)^2$

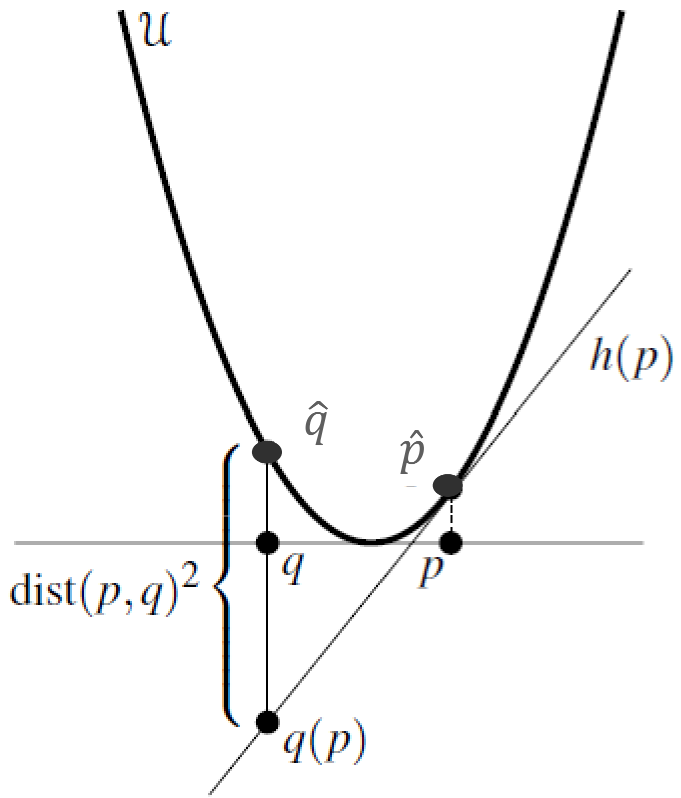
- notice that $\Delta z \geq 0$, and $= 0$ only for $q = p$, which means that $h(p)$ is tangent to U at $\hat{p}$ (and otherwise below U)
- there are no vertical tangent planes to U



The (vertical) distance between $\hat{q}$ and $q(p)$, cont'd

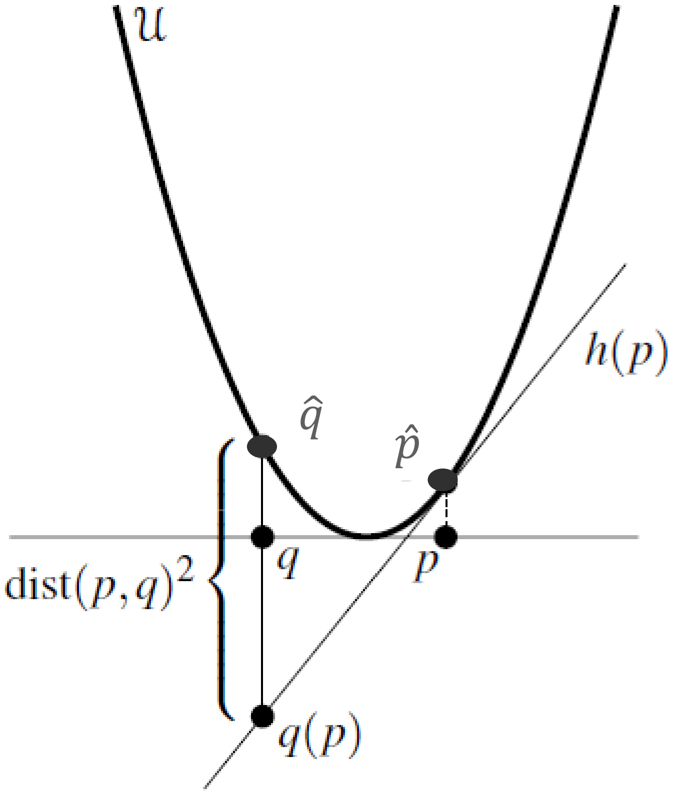
- $h(p)\colon z = 2p_x \mathrm{x} + 2p_y y - ({p_x}^2 + {p_y}^2)$
- $\hat{q}(q_x, q_y, {q_x}^2 + {q_y}^2)$
- $\Delta z = {q_x}^2 + {q_y}^2 - 2p_x q_x - 2p_y q_y + ({p_x}^2 + {p_y}^2) = (q_x - p_x)^2 + (q_y - p_y)^2 = dist(p,q)^2$

- furthermore, the vertical distance between $\hat{q}$ and $h(p)$ encodes the square of the planar distance between p and q



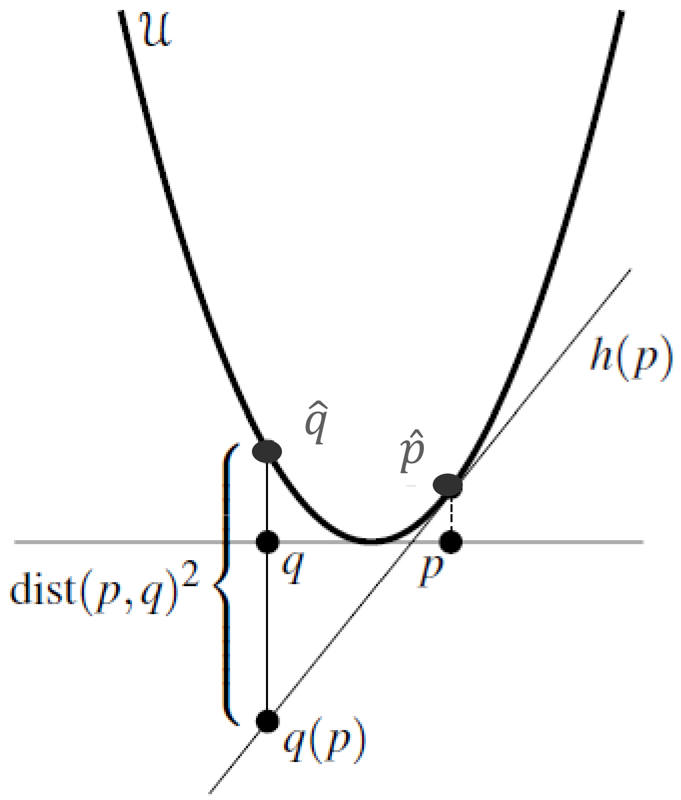
Voronoi diagrams and upper envelopes

- given a set P of n points in the plane
- we produce a plane $h(p)$ for every $p \in P$
- $H := \{h(p) | p \in P\}$
- $UE(H)$ is the upper envelope of the plane in H
- take a point q in the plane, lift it to $\hat{q}$, shoot a vertical ray downward from $\hat{q}$ into $UE(H)$
- the ray will hit the plane $h(p)$, which is vertically closest to $\hat{q}$



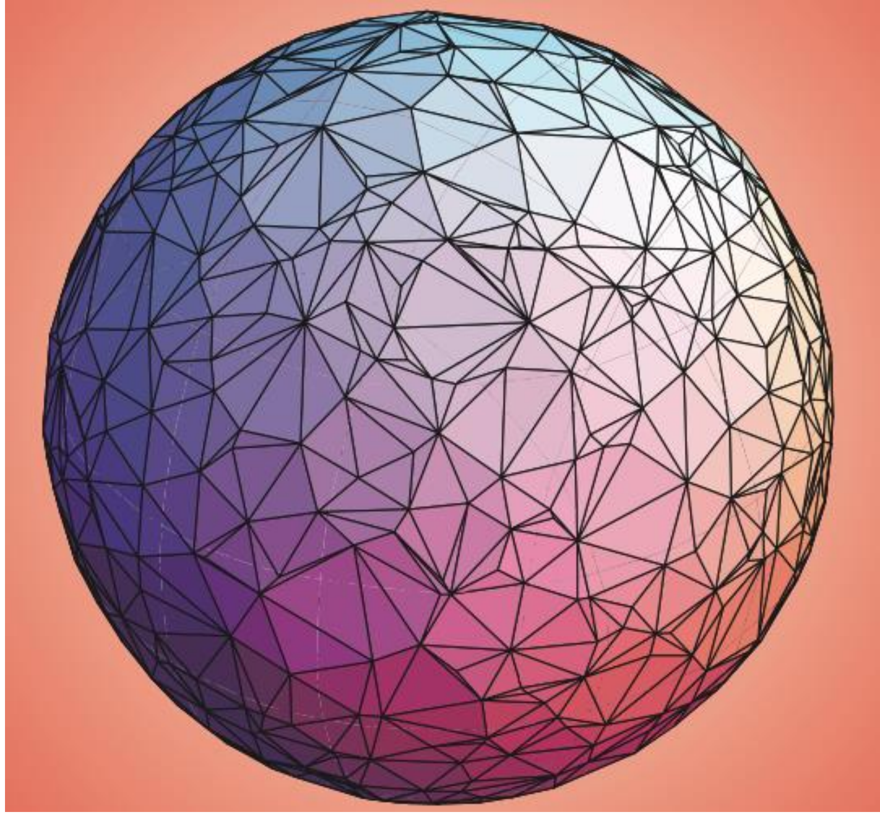
Voronoi diagrams and upper envelopes, cont'd

- the ray will hit the plane $h(p)$, which is vertically closest to $\hat{q}$
- namely, p is the closest point (site) in the plane to q
- claim: the projection onto the plane of $UE(H)$ is the Voronoi diagram of P



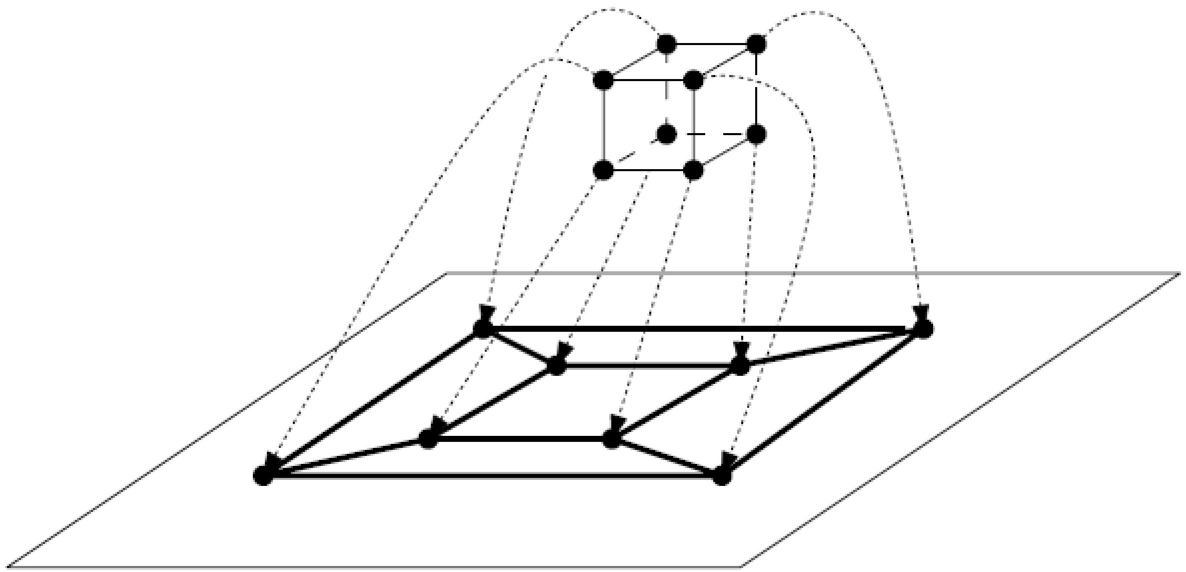
Convex hull in 3D

- the convex hull of a set P of n points in R^3 is a convex polytope whose vertices are points in P
- it therefore has at most n vertices
- its vertices and edges constitute a planar graph
- $CH(P)$ has at most $2n - 4$ faces and at most $3n - 6$ edges



[O'Rourke]

Convex polytopes and planar graphs



- the complexity bounds hold also for non-convex polytopes of *genus* zero with n vertices

Convex hulls in higher dimensions

- the complexity of the convex hull of a set of n points in R^d is $\Theta(n^{\lfloor d/2 \rfloor})$
- it can be computed in $O(n \log n)$ time in R^2 and R^3, and in expected $\Theta(n^{\lfloor d/2 \rfloor})$ time in R^d, for $d > 3$

Connection: Delaunay triangulations and convex hulls in one dimension higher

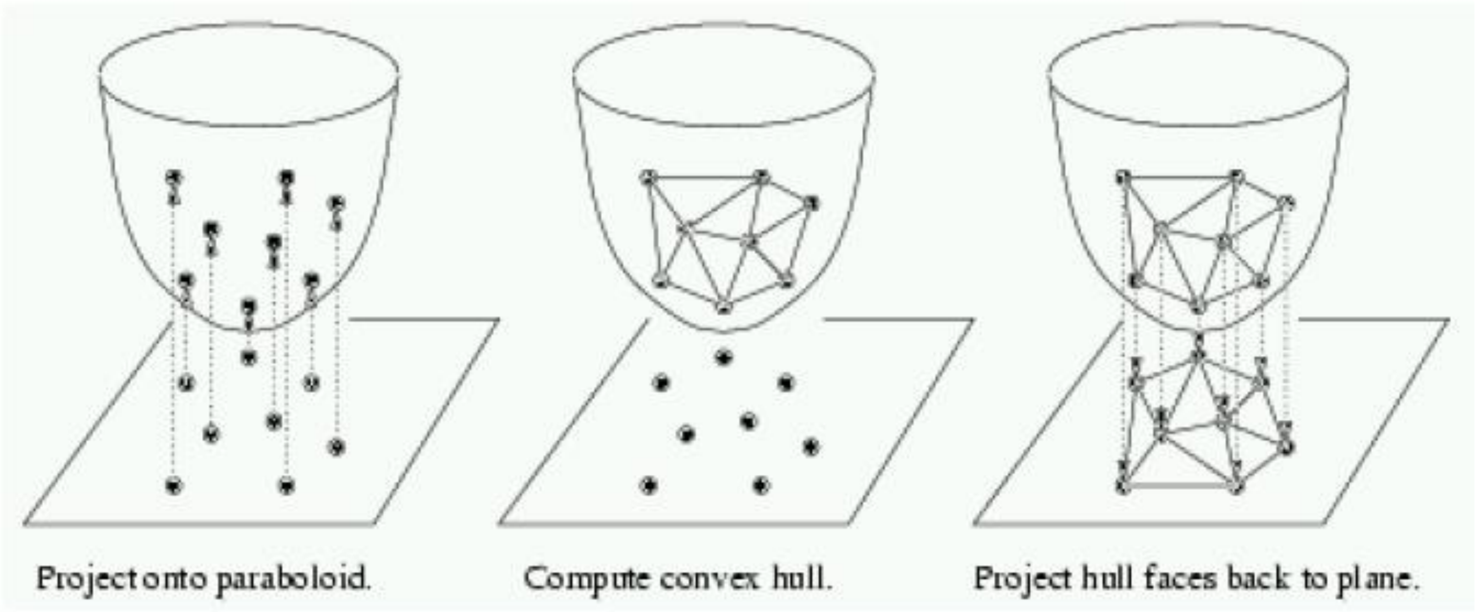
- we are given a set P of points (sites) in general position in the plane
- $\hat{P}$: their projection onto the unit paraboloid U
- $LH(\hat{P})$: the lower convex hull of $\hat{P}$
- consider one facet (triangle, under general position) f of $LH(\hat{P})$, with vertices $\hat{p}, \hat{q}, \hat{r}$
- the projection of the circle $\gamma(p, q, r)$ through p, q, r in the plane onto U lies on the plane $h(f)$ supporting the facet f of the hull, so all other vertices of $\hat{P}$ lie above $h(f)$
- therefore, the circle $\gamma(p, q, r)$ is free of sites of P

Delaunay triangulations and convex hulls, cont'd

- project $LH(\hat{P})$, the lower convex hull of $\hat{P}$, back to the plane
- this projection is a triangulation T of the sites in P
- for every triangle (p, q, r) in T, the circle $\gamma(p, q, r)$ is free of sites of P
- T is the Delaunay triangulation of P

Delaunay triangulations and convex hulls, cont'd

- summary: for a planar set of sites P, the projection onto the plane of $LH(\hat{P})$ is the Delaunay triangulation of P



[Suri]

Summary of connections

Connections

- lower convex hull of points in R^d $\Longleftrightarrow$ upper envelope of hyperplanes in R^d via point-hyperplane duality
- Symmetrically: upper convex hull of points in R^d $\Longleftrightarrow$ lower envelope of hyperplanes in R^d via point-hyperplane duality
- convex hull of points in R^d $\Longleftrightarrow$ intersection of half-spaces in R^d via point-hyperplane duality (through handling subcases)
- Voronoi diagram of points in R^d $\Longleftrightarrow$ upper envelope of hyperplanes in R^{d+1}
- Delaunay triangulation of points in R^d $\Longleftrightarrow$ lower convex hull of points in R^{d+1}

One algorithm?

- an algorithm for computing the convex hull of points in R^2 and R^3, can help us (with a few extra relatively simple procedures) to compute:
  - envelopes in R^2 and R^3
  - intersection of half-spaces in R^2 and R^3
  - Voronoi diagrams of point sites in R^2
  - Delaunay triangulations in R^2

- an algorithm for computing the convex hull of points in any dimension can help us (with a few extra relatively simple procedures) compute these structures in any dimension

THE END

[Jeb Gaither, CGAL arrangements]