Approximations of Set-Valued Functions Based on the Metric Average

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Abstract. This paper investigates the approximation of set-valued functions with compact images (not necessarily convex), by adaptations of the Schoenberg spline operators and the Bernstein polynomial operators. When replacing the sum between numbers in these operators, by the Minkowski sum between sets, the resulting operators approximate only set valued functions with compact-convex images [10]. To obtain operators which approximate set-valued functions with compact images, we use the well known fact that both types of operators for real-valued functions can be evaluated by repeated binary weighted averages, starting from pairs of function values. Replacing the binary weighted averages between numbers by a binary operation between compact sets, introduced in [1] and termed in [4] the "metric average", we obtain operators which are defined for set-valued functions. We prove that the Schoenberg operators so defined approximate set-valued functions which are Hölder continuous, while for the Bernstein operators we prove approximation only for Lipschitz continuous set-valued functions with images in R all of the same topology. Examples illustrating the approximation results are presented.

Key words: Minkowski sum, metric average, set-valued functions, compact sets, Schoenberg spline operators, Bernstein polynomial operators.

1 Introduction

We present in this paper a method for adapting to set-valued functions (multifunctions) certain well known linear positive approximation operators for real-valued functions. We study two types of linear operators, the Schoenberg spline operators and the Bernstein polynomial operators. Both types of operators, when adapted by the usual method of replacing sums between numbers by Minkowski sums of sets, approximate in the Hausdorff metric only multifunctions with compact-convex images [10]. It is shown in [5] that such Bernstein multipolynomials of a set-valued function F with compact images, converge in the Hausdorff metric, with growing degree, to the set-valued function whose images are the convex hulls of the images of F.

Our adaptation method is taken from [4], where the approximation operators were limits of spline subdivision schemes. Here we apply the method successfully to the Schoenberg operators. We use the de Boor algorithm for the evaluation of the Schoenberg operators in terms of repeated binary weighted averages, and replace the binary weighted average between two numbers by a binary operation between sets, introduced in [1], and termed in [4] the "metric average". We prove that with this procedural definition of the Schoenberg operators for multifunctions, the Schoenberg operators approximate a Hölder continuous set-valued function in a rate which equals the Hölder exponent of the multifunction.

For the Bernstein operators we use the de Casteljau algorithm for the evaluation of a Bernstein polynomial in terms of repeated binary weighted averages, and replace the average between two numbers by the metric average of two sets. We prove for F Lipschitz continuous with images in R all of the same topology, that its Bernstein multipolynomial of large enough degree m approximates F with an error bound proportional to $m^{-1/2}$.

The approximation results for both types of operators are illustrated by examples.

We conclude the Introduction by an outline of the paper. In Section 2 we give basic definitions and notations. In particular we discuss the metric average and its relevant properties. In Section 3 the Schoenberg spline operators for real-valued functions are defined, and their evaluation in terms of the de Boor algorithm is briefly reviewed. The procedural definition of the Schoenberg operators for set-valued functions is given in Section 4, together with the approximation results, their proofs and examples. Section 5 discusses the Bernstein polynomials of real-valued functions and their evaluation in terms of the de Casteljau algorithm. In Section 6 the Bernstein operators for set-valued functions are defined, and the proof of the approximation result together with an example are given.

2 Preliminaries

In this section we introduce some definitions and notation. The collection of all nonempty compact subsets of R^n is denoted by $K(R^n)$. By $Co(R^n)$ we denote the collection of all convex sets in $K(R^n)$, and by coA we denote the convex hull of A. The Euclidean distance from a point a to a set

$B \in K(R^n)$ is defined as

$$\operatorname{dist}(a, B) = \inf_{b \in B} |a - b|,$$

where $|\cdot|$ is the Euclidean norm in R^n.
The Hausdorff distance between two sets $A, B \in K(R^n)$ is defined by

$$\operatorname{haus}(A, B) = \max \left\{ \sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A) \right\}.$$

The set of all projections of $a \in R^n$ into a set $B \in K(R^n)$ is

$$\Pi_B(a) = \{ b \in B : |a - b| = \operatorname{dist}(a, B) \}.$$

For $A, B \in K(R^n)$ the projection of A on B is the set

$$\Pi_B(A) = \{ \Pi_B(a) : a \in A \}.$$

A linear Minkowski combination of two sets A and B from $K(R^n)$ is

$$\lambda A + \mu B = \{ \lambda a + \mu b, a \in A, b \in B \},$$

with $\lambda, \mu \in R$. The Minkowski sum corresponds to a linear Minkowski combination with $\lambda = \mu = 1$.

Definition 2.1. Let $A, B \in K(R^n)$ and $0 \le t \le 1$. The t-weighted metric average of A and B is

$$A \oplus_t B = \{ ta + (1-t)\Pi_B(a) : a \in A \} \bigcup \{ t\,\Pi_A(b) + (1-t)b : b \in B \} \quad (1)$$

The most important properties of the metric average are presented below [4] :
For $A, B \in K(R^n)$ and $0 \le t \le 1$, $0 \le s \le 1$

1. $A \oplus_0 B = B, \quad A \oplus_1 B = A, \quad A \oplus_t B = B \oplus_{1-t} A$
2. $A \oplus_t A = A$
3. $A \bigcap B \subseteq A \oplus_t B \subseteq tA + (1-t)B \subseteq co(A \bigcup B)$
4. $\operatorname{haus}(A \oplus_t B, A \oplus_s B) = |t - s| \operatorname{haus}(A, B)$
5. $A \oplus_t B = tA + (1-t)B$, $A, B \in Co(R)$

It follows from properties 1 and 4 that

$$\operatorname{haus}(A \oplus_t B, A) = (1-t)\operatorname{haus}(A, B), \quad \operatorname{haus}(A \oplus_t B, B) = t \operatorname{haus}(A, B) \quad (2)$$

3 Schoenberg operators for real-valued functions, and their evaluation by repeated binary averages

The m-th order Schoenberg spline operator (Schoenberg's variation diminishing spline approximation) $S_m f$ to a continuous function f on R is given by

$$S_m f = \sum_{i \in Z} f(i) B_m (\cdot - i) \, ,$$

where $B_m(t)$ is the B-spline of order m with integer knots and support $[0, m]$ [3]. For the knot sequence hZ, with small h, we consider the operator

$$S_{m,h} f = \sum_{i \in Z} f(ih) B_m \left(\frac{\cdot}{h} - i \right) . \tag{3}$$

For $f \in C(R)$ $\lim_{h \to 0} S_{m,h} f(t) = f(t)$ $t \in R$ [3].

$S_{m,h} f$ can be evaluated by an algorithm (known as the de Boor algorithm) for the computation of a spline function given in terms of the B-spline basis, based on the recurrence formula for B-splines.

For $j \leq t < j+1$, (3) can be written as

$$S_{m,h} f(th) = \sum_{i=j-m+k+1}^{j} a_i^k B_{m-k} (t - i) \, , \tag{4}$$

with $0 \leq k \leq m-1$ and

$$a_i^0 = f(ih), \quad i = j-m+1, ..., j$$

$$a_i^k = \frac{i+m-k-t}{m-k} \, a_{i-1}^{k-1} + \frac{t-i}{m-k} \, a_i^{k-1}, \quad i = j-m+k+1, ..., j \, . \tag{5}$$

Introducing the notation

$$\lambda_i^k = \frac{i+m-k-t}{m-k}, \quad i = j-m+k+1, ..., j \, , \; k = 1, ..., m-1 \, , \tag{6}$$

we observe that a_i^k is a convex combination of a_{i-1}^{k-1} and a_i^{k-1} with coefficients λ_i^k, $1 - \lambda_i^k$. The case $k = m-1$ yields

$$S_{m,h} f(th) = a_j^{m-1}. \tag{7}$$

Remark 3.1. It follows from (4) with $k = 0$ that $S_{m,h}f(th)$ at $t \in [j, j+1)$ depends only on $f(ih)$ $i = j - m + 1, ..., j$. A better approximation is the symmetric Schoenberg operator:

$$\widetilde{S}_{m,h}f = \sum_{i \in Z} f(ih)\widetilde{B}_m\left(\frac{\cdot}{h} - i\right), \quad \text{where } \widetilde{B}_m(t) = B_m\left(t - \frac{m}{2}\right) \tag{8}$$

For $t \in [j, j+1)$ $\widetilde{S}_{m,h}f(th)$ is a convex combination of values of f at a set of symmetric points relative to $(jh, (j+1)h)$. For even m the evaluation of $\widetilde{S}_{m,h}f$ is similar to that of $S_{m,h}f$.
In this work we study the operator $S_{m,h}$ for set-valued functions.

4 Schoenberg operators for set-valued functions

Let $F : R \to K(R^n)$ be a set-valued function. We define the set-valued Schoenberg operator of order m in terms of its evaluation according to the de Boor algorithm, using the metric average as the basic binary operation and the initial sets $\{F_i^0 = F(i), i \in Z\}$. To calculate the spline operator $S_{m,h}F(th)$ at $t \in [j, j+1)$ we use an extension of (5) and (7) with the average between two numbers replaced by the metric average of two sets. Thus for $k = 1, ..., m-1$ we define recursively the sets

$$F_i^k = F_{i-1}^{k-1} \oplus_{\lambda_i^k} F_i^{k-1}, \tag{9}$$

with λ_i^k given by (6) and as in (7), determine $S_{m,h}F(th)$ to be

$$S_{m,h}F(th) = F_j^{m-1}. \tag{10}$$

First we prove some basic results, which are used in the proof of the approximation theorem.

Lemma 4.1. *Given an initial sequence of compact sets $\{F_i^0, i \in Z\} \subset K(R^n)$, we define the sets at level k by repeated application of (9). Let*

$$d^k = \sup_{i \in Z} \mathrm{haus(F^k_{i-1}, F^k_i)}. \tag{11}$$

Then

$$d^k \leq \frac{m-k-1}{m-1} d^0, \; k = 1, ..., m-2.$$

Proof. It follows from (9) and (2) that

$$\begin{aligned}\mathrm{haus}(F_i^k, F_i^{k-1}) &= \mathrm{haus}(F_{i-1}^{k-1} \oplus_{\lambda_i^k} F_i^{k-1}, F_i^{k-1}) \\ &= \lambda_i^k \,\mathrm{haus}(F_{i-1}^{k-1}, F_i^{k-1}) \leq \frac{i+m-k-t}{m-k}\, d^{k-1}.\end{aligned}$$

Thus

$$\mathrm{haus}(F_i^k, F_i^{k-1}) \leq \frac{i+m-k-t}{m-k}\, d^{k-1}. \tag{12}$$

In the same way we obtain

$$\begin{aligned}\mathrm{haus}(F_i^{k-1}, F_{i+1}^k) &= \mathrm{haus}(F_i^{k-1}, F_i^{k-1} \oplus_{\lambda_{i+1}^k} F_{i+1}^{k-1}) \\ &= (1-\lambda_{i+1}^k)\mathrm{haus}(F_i^{k-1}, F_{i+1}^{k-1}) \leq \frac{t-i-1}{m-k}\, d^{k-1}.\end{aligned}$$

Therefore

$$\mathrm{haus}(F_i^{k-1}, F_{i+1}^k) \leq \frac{t-i-1}{m-k}\, d^{k-1}. \tag{13}$$

By the triangle inequality and using the estimates (12) and (13) we get:

$$\mathrm{haus}(F_i^k, F_{i+1}^k) \leq \mathrm{haus}(F_i^{k-1}, F_i^k) + \mathrm{haus}(F_i^{k-1}, F_{i+1}^k) \leq \frac{m-k-1}{m-k}\, d^{k-1}.$$

This leads to

$$d^k \leq \frac{m-k-1}{m-k}\, d^{k-1}. \tag{14}$$

Now, using (14) repeatedly, we obtain the claim of the lemma

$$\begin{aligned}d^k &\leq \frac{m-k-1}{m-k}\, d^{k-1} \leq \frac{m-k-1}{m-k} \cdot \frac{m-(k-1)-1}{m-(k-1)} \cdots \frac{m-2-1}{m-2} \cdot \frac{m-1-1}{m-1}\, d^0 \\ &= \frac{m-k-1}{m-1}\, d^0.\end{aligned}$$

$\square$

Lemma 4.2. *Let $S_{m,h}F$, be define by (9) and (10). Then for any point $t \in [j, j+1)$*

$$\mathrm{haus}(\, S_{m,h}F(th), F_j^0\,) \leq d^0 \frac{m}{2}. \tag{15}$$

Proof. By (10), the triangle inequality, (12) and Lemma 4.1

$$\text{haus}(S_{m,h}F(th), F_j^0) = \text{haus}(F_j^{m-1}, F_j^0) \le \sum_{k=1}^{m-1} \text{haus}(F_j^{k-1}, F_j^k) \le \sum_{k=1}^{m-1} \frac{m-k+j-t}{m-k}\, d^{k-1}$$

$$\le \sum_{k=1}^{m-1} \frac{m-k+j-t}{m-k} \cdot \frac{m-(k-1)-1}{m-1}\, d^0 = \frac{d^0}{m-1} \sum_{k=1}^{m-1} (m-k+j-t)$$

$$= d^0 \left(m + j - t - \frac{1}{m-1} \sum_{k=1}^{m-1} k \right) = d^0 \left(m + j - t - \frac{m}{2} \right).$$

Finally we obtain

$$\text{haus}(S_{m,h}F(th), F_j^0) \le d^0 \left(\frac{m}{2} + j - t \right) \le d^0 \frac{m}{2}.$$

□

As a consequence of the last lemma, we get the approximation result.

Theorem 4.1. *Let the set-valued function* $F : [0,1] \to K(R^n)$ *be Hölder continuous with exponent* $\nu \in (0,1]$,

$$\text{haus}(F(x), F(z)) \le C_\nu |x - z|^\nu, \quad x, z \in [0,1].$$

Let $F_i^0 = F(ih)$, $i = 0, 1, ..., N$ *with* $hN = 1$, *and* $F_i^0 = \{0\}$ *otherwise. Then for any* $x \in [h(m-1), 1]$

$$\text{haus}(S_{m,h}F(x), F(x)) \le \left(\frac{m}{2} + 1 \right) C_\nu h^\nu. \tag{16}$$

Proof. For $x \in [(m-1)h, 1]$, let $l_x \in Z$ be such that $x \in [l_x h, (l_x + 1)h)$. Note that for such x, the value $S_{m,h}F(x)$ depends on values F_i^0 for $i \in \{l_x - m + 1, l_x - m + 2, ..., l_x\} \subset \{0, 1, ..., N\}$.
By the triangle inequality we have

$$\text{haus}(S_{m,h}F(x), F(x)) \le \text{haus}(S_{m,h}F(x), F_{l_x}^0) + \text{haus}(F_{l_x}^0, F(x))$$

Hence by Lemma 4.2 we obtain

$$\text{haus}(S_{m,h}F(x), F(x)) \le d^0 \frac{m}{2} + \text{haus}(F_{l_x}^0, F(x)) \tag{17}$$

Now, by the Hölder continuity of F,

$$d^0 \le C_\nu h^\nu$$

and

$$\text{haus}(F_{l_x}^0, F(x)) = \text{haus}(F(l_x h), F(x)) \le C_\nu h^\nu$$

This together with (17) leads to the claim of the theorem. □

Example 4.1. We construct Schoenberg approximations to the multifunction $F(x)$ defined by

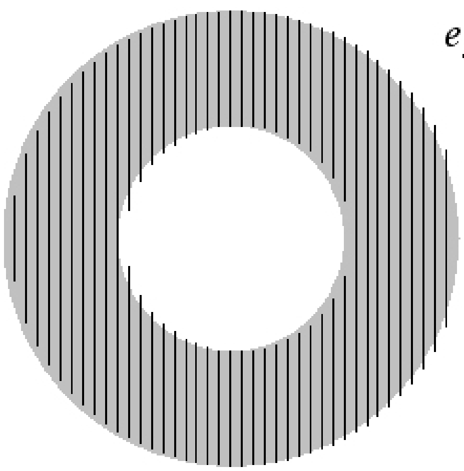
$$F(x) = \left\{y : \max\{0, (r/2)^2 - (x-0.5)^2\} \le y^2 \le r^2 - (x-0.5)^2\right\}, \quad r = 0.5,\ x \in [0,1]. \tag{18}$$

(a) Approximation with $S_{3,h}F$.

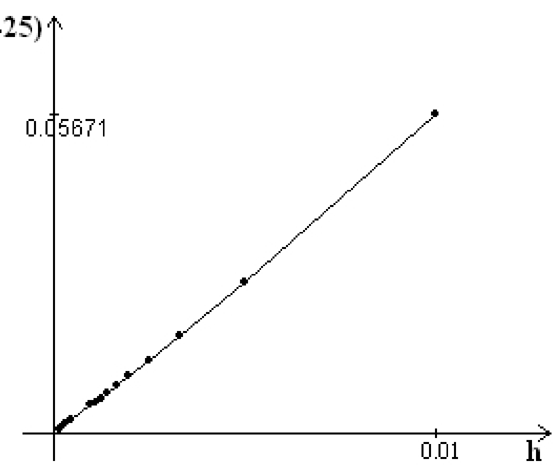
The original set-valued function is presented in gray on the left-hand side of Figure 4.1, 40 cross-sections of the reconstructed shape, $S_{3,0.01}F$, is depicted in black. The graph of

$$e_h(x) = \mathrm{haus}(S_{3,h}F(x), F(x))$$

at $x = 0.425$ as function of h, is shown on the right-hand side of Figure 4.1.



(a)



(b)

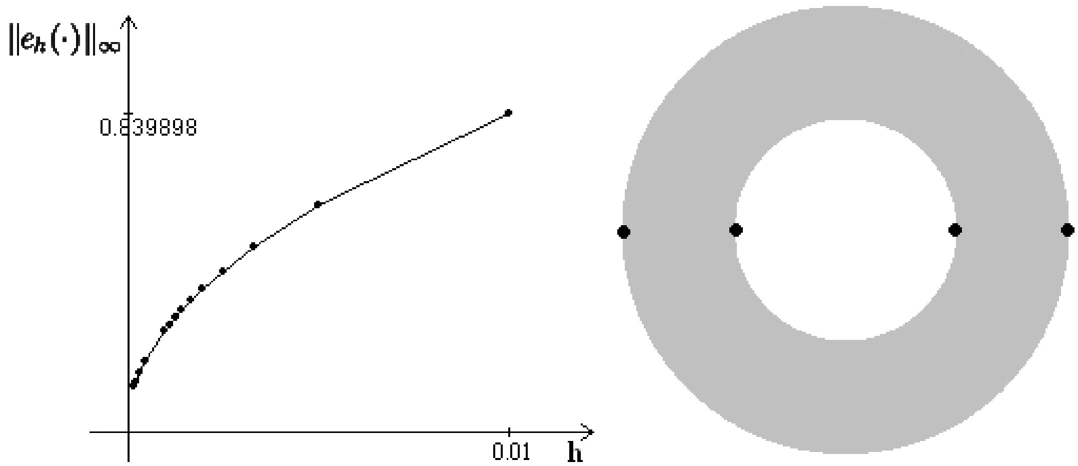
Figure 4.1.

(a) F - in gray. Forty cross-sections of $S_{3,0.01}F$ - in black.
(b) Error between the original and the reconstructed cross-sections at $x = 0.425$ as function of h.

We note that $e_h(0.425)$ changes almost linearly with h. This is in accordance with Theorem 4.1, since at $x = 0.425$ F is Lipschitz continuous $(\nu = 1)$.

The graph of the maximal error between cross-sections of the reconstructed shape, $S_{3,h}F$ and the corresponding cross-sections of (18) as a function of h is presented in Figure 4.2 (a). The maximal error is obtained at the points of change of topology of the cross-sections of (18), which are depicted in Figure 4.2 (b).

To verify that the decay of the error in this figure is in accordance with Theorem 4.1, we show that F in (18) is Hölder continuous with exponent $1/2$, at points of change of topology.



(a) (b)

Figure 4.2.

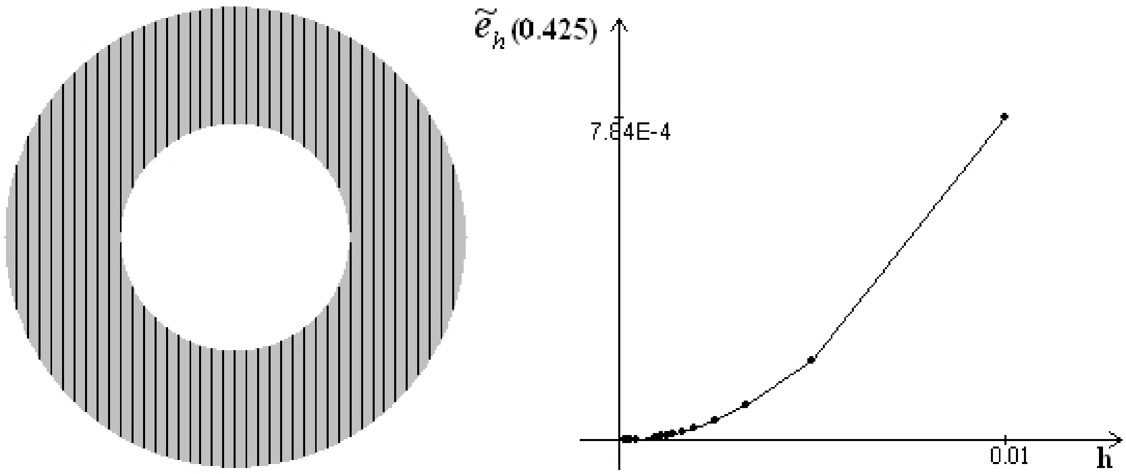
(a) Maximal error between the original and the reconstructed cross-sections as function of h.
(b) Points of change of topology, where the maximal error is attained.

Consider the boundary of the ring in 2D determined by (18). Locally near the points of change of topology of cross-sections the boundary can be described by a scalar function $y = f(x)$, or by $x = g(y)$. One can see easily that the derivative of f tends to infinity at points of change of topology (see Figure 4.2 (b)). Let $x = g(y)$ be the inverse function of f and let (x_0, y_0) be a point of topology change. Since $g'(y_0) = 1/f'(x_0) = 0,$ we get by the Taylor expansion of degree 2 of $g(y)$ about y_0,

$$g(y) - g(y_0) = (\Delta y)^2 \cdot \frac{g''(y_0)}{2} + R_3, \quad \Delta y = y - y_0.$$

Thus for $|x - x_0|$ =h and since $|R_3|/|\Delta y|^2 = o(|\Delta y|)$, we obtain $\Delta y \approx \sqrt{2h/g''(y_0)}$, from which it can be concluded that F is Hölder continuous with exponent $1/2$ at the points of change of topology.

(b) Approximation with $\widetilde{S}_{4,h}F$.



(a) (b)

Figure 4.3.

(a) F - in gray. Forty cross-sections of $\widetilde{S}_{4,\,0.01}F$ - in black.
(b) Error between the original and the reconstructed cross-sections at $x = 0.425$ as function of h.

Figure 4.3 is similar to Figure 4.1 but with $\widetilde{S}_{4,h}F$ replacing $S_{3,h}F$. It is easy to observe that the behavior of the error function is almost quadratic in h. We conjecture that F is smooth enough at $x = 0.425$ in a sense yet to be defined, and that

$$\widetilde{e}_h(x) = \mathrm{haus}(F(x), \widetilde{S}_{4,h}F(x)) = O(h^2), \tag{19}$$

in points of smoothness of F. Moreover, we conjecture that (19) holds for $\widetilde{S}_{2\widetilde{m},h}F$ for all $\widetilde{m} \geq 2$. This is an improvement over the approximation rate in Theorem 4.1, as in the case of real-valued functions.

5 Bernstein polynomials of real-valued functions and their evaluation by repeated binary averages.

For $f \in C[0,1]$, the Bernstein polynomial of degree m is

$$B_m(f,u) = \sum_{i=0}^{m} \binom{m}{i} u^i (1-u)^{m-i} f\left(\frac{i}{m}\right). \tag{20}$$

The value $B_m(f,u)$ can be calculated recursively by using the de Casteljau algorithm [9] in terms of repeated binary averages. The algorithm is based on the following recurrence relation,

$$B_{i,m}(u) = (1-u)B_{i,m-1}(u) + u\, B_{i-1,m-1}(u), \tag{21}$$

where $B_{i,m}(u) = \binom{m}{i} u^i (1-u)^{m-i}$.

$B_m(f,u)$ in (20) for $u \in [0,1]$ can be presented by a repeated application of (21) as:

$$B_m(f,u) = \sum_{i=0}^{m} \binom{m}{i} u^i (1-u)^{m-i} f_i^0 = \sum_{i=0}^{m-k} \binom{m-k}{i} u^i (1-u)^{m-k-i} f_i^k, \tag{22}$$

with the values f_i^k given recursively by

$$f_i^k = (1-u) f_i^{k-1} + u\, f_{i+1}^{k-1}, \quad i = 0,1,...,m-k,\ k = 1,...,m, \tag{23}$$

and with $f_i^0 = f(i/m),\ i = 0,1,...,m.$

Comparing formulas (23) with formulas (5) one can easily see that the de Boor algorithm is a generalization of the de Casteljau algorithm.

Taking $k = m$ in (22) we obtain $B_m(f,u) = f_0^m$. Thus the Bernstein polynomial of a real-valued function can be defined by repeated binary averages.

6 Bernstein operators for set-valued functions

Let $F : [0,1] \to K(R^n)$ be a set-valued function with compact images. Let $F_i^0 = F(i/m)$ be the initial cross-sections, $F_i^0 \in K(R^n),\ i = 0,1,...,m.$ Consider the Bernstein polynomial of a set-valued function, having the form of the Bernstein polynomial of a real-valued function with sums of numbers replaced by Minkowski sums of sets,

$$B_m^M(F,u) = \sum_{i=0}^{m} \binom{m}{i} u^i (1-u)^{m-i} F\left(\frac{i}{m}\right) \tag{24}$$

It is shown in [5] that the limit of $B_m^M(F,u)$, for a fixed $u \in (0,1)$, when $m \to \infty$, is the convex hull of $F(u)$. Therefore, the set-valued polynomial (24) is a good approximation for functions with convex compact images. To obtain an operator, which does not convexify the initial data, we define constructively the Bernstein approximation of F in terms of the de Casteljau algorithm with the metric average as the basic binary operation. Thus to calculate the value of the Bernstein polynomial of degree m at the point $u \in [0,1]$, $B_m(F,u)$, we use the following extension of (23):

$$F_i^k = F_i^{k-1} \oplus_{1-u} F_{i+1}^{k-1}, \quad i = 0,1,...,m-k, \ k = 1,...,m \tag{25}$$

and define

$$B_m(F,u) = F_0^m. \tag{26}$$

First we show,

Lemma 6.1. *Let* $F^k = \{F_i^k, \ i = 0,...,m-k\}$ *be define as above, and let*

$$d^k = \sup_{i \in Z \bigcap [1,m-k]} \mathrm{haus}(\mathrm{F_{i-1}^k}, \mathrm{F_i^k}), \quad \mathrm{k = 0,1,...,m-1}. \tag{27}$$

Then

$$d^k \leq d^0, \quad k = 1,...,m-1.$$

Proof. From (25) and (2)

$$\begin{aligned} \mathrm{haus}(F_i^k, F_i^{k-1}) &= \mathrm{haus}(F_i^{k-1}, F_i^{k-1} \oplus_{1-u} F_{i+1}^{k-1}) \\ &= u \, \mathrm{haus}(F_i^{k-1}, F_{i+1}^{k-1}) \leq u \, d^{k-1}. \end{aligned} \tag{28}$$

In the same way we obtain

$$\begin{aligned} \mathrm{haus}(F_i^{k-1}, F_{i-1}^k) &= \mathrm{haus}(F_{i-1}^{k-1} \oplus_{1-u} F_i^{k-1}, F_i^{k-1}) \\ &= (1-u) \, \mathrm{haus}(F_{i-1}^{k-1}, F_i^{k-1}) \leq (1-u) \, d^{k-1}. \end{aligned} \tag{29}$$

Now, by the triangle inequality, (28) and (29) we get,

$$\begin{aligned} \mathrm{haus}(F_{i-1}^k, F_i^k) &\leq \mathrm{haus}(F_i^{k-1}, F_{i-1}^k) + \mathrm{haus}(F_i^{k-1}, F_i^k) \\ &\leq (1-u) d^{k-1} + u \, d^{k-1} = d^{k-1}. \end{aligned}$$

Thus

$$d^k \leq d^{k-1},$$

which implies the claim of the lemma. ∎

We do not have a proof of the convergence of $B_m(F,u)$ to $F(u)$ as $m\to\infty$. Yet we have a proof in the case of set-valued functions with cross-sections in R all of the same topology. Our proof is based on the following result from [10]:

Result 6.1. For $F:[0,1]\to Co(R^n)$ Lipschitz continuous

$$\mathrm{haus}(\mathrm{B^M_m(F,u),F(u)})\leq \mathrm{C}/\sqrt{\mathrm{m}},\quad \mathrm{u}\in[0,1],$$

where $B^M_m(F,u)$ is defined by (24) and the constant C depends only on the Lipschitz constant of F.

Any set A in R consists of a number of disjoint intervals, some possibly with empty interior. Thus A can be written in the form $A=\bigcup_{j=1}^J A_j$ with A_j , $j=1,...,J$ ordered and disjoint intervals, namely $a_j<a_{j+1}$ for any $a_j\in A_j$ and $a_{j+1}\in A_{j+1}$, $j=1,...,J-1$. We denote this by $A_1<...<A_J$. We introduce a measure of separation of such a set with $J>1$:

$$s(A)=\inf_{l,j\in\{1,...,J\},l\neq j}\{\mathrm{dist}(a,A_j):a\in A_l\} \tag{30}$$

In the following we assume that J is finite. We discuss only the case $J>1$, since $J=1$ is a special case of Result (6.1).

Definition 6.1. Two sets $A,B\in K(R)$ are called topologically equivalent if each is a union of the same number of disjoint intervals, namely

$$A=\bigcup_{j=1}^J A_j,\quad B=\bigcup_{j=1}^J B_j, \tag{31}$$

with A_j , $j=1,...,J$ and B_j , $j=1,...,J$ disjoint ordered intervals.

Definition 6.2. Let $A,B\in K(R)$ be topologically equivalent. The sets A,B are called metrically equivalent if

$$\Pi_B(A_j)\subset B_j\quad\text{and}\quad \Pi_A(B_j)\subset A_j,\quad j=1,...,J. \tag{32}$$

This relation between the two sets is denoted by $A\sim B$.

Lemma 6.2. *Let $A,B\in K(R)$ be topologically equivalent. If*

$$\mathrm{haus(A,B)}<\frac{\mathrm{min(s(A),s(B))}}{2} \tag{33}$$

then A and B are metrically equivalent.

Proof. Assume the opposite, i.e. that (33) holds, but A, B are not metrically equivalent, namely there exists a subset $B_l \in B$ such that two points from B_l have their closest points in A in two subsets of A, say A_j and A_{j+1}. By the continuity of the projection mapping there exists a point $\widetilde{b} \in B_l$ such that $\{a_1, a_2\} \subset \Pi_A(\widetilde{b})$, $a_1 \in A_j$, $a_2 \in A_{j+1}$.
By the triangle inequality,

$$\operatorname{dist}(a_1, a_2) \leq \operatorname{dist}(\widetilde{b}, a_1) + \operatorname{dist}(\widetilde{b}, a_2) = 2 \operatorname{dist}(\widetilde{b}, A). \tag{34}$$

Now, by the definition of the Hausdorff distance, (34) and (30) we obtain:

$$\operatorname{haus}(A, B) \geq \operatorname{dist}(\widetilde{b}, A) \geq \frac{1}{2}\operatorname{dist}(a_1, a_2) \geq \frac{1}{2}s(A)$$

in contradiction to assumption (33). Thus $\Pi_A(B_l) \subset A_j$, and by symmetry $\Pi_B(A_j) \subset B_k$. It remains to prove that $k = l$.

Let $a \in A_j$ and $b_l \in B_l$ be such that $a \in \Pi_A(b_l)$. Let $b_k \in B_k$ be such that $b_k \in \Pi_B(a)$. By the triangle inequality and by the definition of the Hausdorff distance

$$\begin{aligned} \operatorname{dist}(b_l, b_k) &\leq \operatorname{dist}(b_l, a) + \operatorname{dist}(a, b_k) \\ &\leq \operatorname{dist}(b_l, A) + \operatorname{dist}(a, B) \leq 2 \operatorname{haus}(A, B). \end{aligned} \tag{35}$$

Now by (30) we have if $k \neq l$ that

$$s(B) \leq \operatorname{dist}(b_l, b_k).$$

This together with (35) contradicts (33). Hence $\Pi_A(B_l) \subset A_j$ and $\Pi_B(A_j) \subset B_l$. Since A and B are both of the form (31), we conclude that $l = j$. Thus $A \sim B$. ☐

Corollary 6.1. *The metric average of two topologically equivalent sets A and B, satisfying (33), is given by*

$$A \oplus_t B = \bigcup_{j=1}^{J} A_j \oplus_t B_j. \tag{36}$$

Lemma 6.3. *Let $\{F_i^0 \subset R,\ i = 0, 1, ..., m\}$ be topologically equivalent, of the form*

$$F_i^0 = \bigcup_{j=1}^{J} F_{i,j}^0,$$

with $F^0_{i,j}\,, j=1,...,J$ *disjoint ordered intervals. Define* $\{F^k_i\}$ *and* d^k *by (25) and (27) respectively, and define*

$$s^k = \min\{s(F^k_i) : i=0,1,...,m-k\},\ k=0,1,...,m-1. \tag{37}$$

If $d^0 < s^0/2$ *, then*

$$d^k < s^k/2,\ k=1,...,m-1, \tag{38}$$

and the sets $\{F^k_i,\ i=0,...,m-k,\ k=0,...,m\}$ *are topologically equivalent.*

Proof. We prove the lemma by induction. We assume that the sets $\{F^l_i : i=0,...,m-l\,, l=0,...,k-1\}$ are topologically equivalent and that $d^{k-1} \leq s^{k-1}/2$. Note that the induction hypothesis is satisfied for $k=1$. Since two consecutive sets of $\{F^{k-1}_i\}$ are metrically equivalent, by the induction hypothesis and by Lemma 6.2, we get by Corollary 6.1 that

$$F^k_i = \bigcup_{j=1}^{J} F^{k-1}_{i,j} \oplus_{1-u} F^{k-1}_{i+1,j}\,,\ i=0,...,m-k. \tag{39}$$

Now, by property 5 of the metric average (see Section 2)

$$F^{k-1}_{i,j} \oplus_{1-u} F^{k-1}_{i+1,j} = (1-u)F^{k-1}_{i,j} + u\,F^{k-1}_{i+1,j} = I^k_{i,j}, \tag{40}$$

where $I^k_{i,j}$ is an interval.
First we show that

$$\sigma^k_i = \min\{|c_1-c_2| : c_1 \in I^k_{i,j},\ c_2 \in I^k_{i,j+1},\ j\in\{1,...,J-1\}\} \geq s^{k-1} \tag{41}$$

Let $\sigma^k_i = |c^*_1 - c^*_2|$. By (39) and (40), we have

$$c^*_l = (1-u)a_l + u\,b_l,\ l=1,2$$

with $a_1 \in F^{k-1}_{i,j}$, $b_1 \in F^{k-1}_{i+1,j}$ and $a_2 \in F^{k-1}_{i,j+1}$, $b_2 \in F^{k-1}_{i+1,j+1}$ for some $j \in \{1,...,J-1\}$. Thus,

$$|c^*_1 - c^*_2| = |(1-u)(a_1-a_2) + u(b_1-b_2)|.$$

Since the differences (a_1-a_2) and (b_1-b_2) have the same sign, we can write $|c^*_1-c^*_2| = (1-u)|a_1-a_2| + u|b_1-b_2)|$. Finally, using (30) we obtain:

$$\begin{aligned}\sigma^k_i &= |c^*_1-c^*_2| = (1-u)|a_1-a_2| + u|b_1-b_2| \\ &\geq (1-u)\,s(F^{k-1}_i) + u\,s(F^{k-1}_{i+1}) \geq \min(s(F^{k-1}_i), s(F^{k-1}_{i+1})) \geq s^{k-1}\end{aligned}$$

It follows from (41) that $I^k_{i,j} \bigcap I^k_{i,j+1} = \emptyset$ for $j\in\{1,...,J-1\}$, and in view of (39) and (40) we conclude that F^k_i is topologically equivalent to F^{k-1}_i, F^{k-1}_{i+1}.

Moreover by (41) $s^k = \min_i \sigma_i^k$, and

$$s^k \geq s^{k-1}.$$

This together with Lemma 6.1 and the induction hypothesis leads to

$$d^k \leq d^{k-1} < s^{k-1}/2 \leq s^k/2.$$

Thus the induction hypothesis holds for k which concludes the proof of the lemma. □

Lemmas 6.3 and 6.2 lead to

Corollary 6.2. *Let the sets* $\{F_i^k : i = 0, ..., m-k\,, k = 0, ..., m-1\}$ *be as in Lemma 6.3. Then* $F_i^k \sim F_{i-1}^k\,, i = 1, ..., m-k\,, k = 0, ..., m-1$.

Now we can prove,

Theorem 6.1. *Let the set-valued function* $F : [0,1] \to K(R)$ *be Lipschitz continuous, such that for each* $t \in [0,1]$, $F(t) = \bigcup_{j=1}^J F_j(t)$, *with* $J > 1$, *where* $\{F_j(t)\}$ *are disjoint ordered intervals. Then for* m *large enough*

$$\mathrm{haus}(\mathrm{B_m(F,u),F(u)}) \leq \widetilde{\mathrm{C}}/\sqrt{\mathrm{m}}, \quad \mathrm{u} \in [0,1]. \tag{42}$$

Proof. Let m be such that for $F_i^0 = F(i/m)$, $i = 0, ..., m$,

$$d^0 < s^*/2,$$

with d^0 defined by (27) and

$$s^* = \inf_{0 \leq t \leq 1} s(F(t)) > 0.$$

Such m exists since F is Lipschitz continuous. In fact m has to be large enough. Obviously $s^* \leq s^0$, where s^0 is defined in (37). Thus $d^0 \leq s^0/2$. Now, by Corollary 6.1 and Property 5 of the metric average we get

$$B_m(F,u) = \bigcup_{j=1}^J B_m^M(F_j,u).$$

Therefore

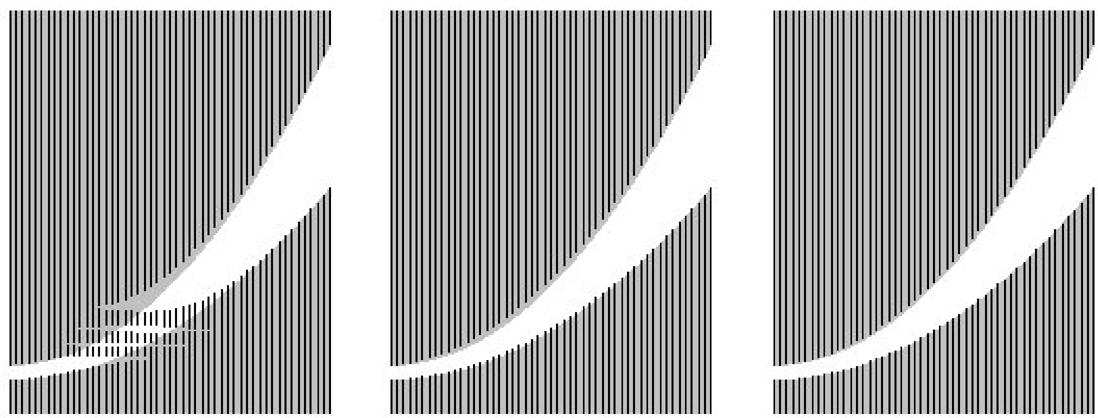
$$\mathrm{haus}(F(u), B_m(F,u)) = \max_{1 \leq j \leq J} \mathrm{haus}(F_j(u), B_m^M(F_j,u)),$$

and (42) follows from Result 6.1. □

Example 6.1. To illustrate Theorem 6.1, we consider the function $F(x)$ defined by

$$F(x) = \left\{ \{y : 1 \le y \le 0.06x^2 + 2\} \bigcup \right. \\ \left. \{y : 0.1x^2 + 2.5 \le y \le 13.5\} \right\}, \; x \in [0, 10]. \tag{43}$$

This function is depicted in gray in (a), (b), (c) of Figure 6.1. Fifty cross-sections of the reconstructed shapes, $B_{12}(F,u)$, $B_{13}(F,u)$ and $B_{30}(F,u)$, are colored by black and presented in (a), (b) and (c) of Figure 6.1 respectively. Note that (33) does not hold for $m = 12$, while for $m = 13$ and $m = 30$ (33) holds. Figure 6.1 shows that for $m = 12$ there is no approximation, while $B_{13}(F,u)$ is already approximating the shape. The approximation by $B_{30}(F,u)$ is better than that by $B_{13}(F,u)$.



(a) (b) (c)

Figure 6.1.

(a) $F(x)$ - in gray. Fifty cross-sections of $B_{12}(F,u)$ - in black.
(b) $F(x)$ - in gray. Fifty cross-sections of $B_{13}(F,u)$ - in black.
(a) $F(x)$ - in gray. Fifty cross-sections of $B_{30}(F,u)$ - in black.

7 Conclusion

We expect that the approximation methods studied in this paper will become useful for practical applications. For this, an effective algorithm for the evaluation of the metric average is needed. An algorithm for computing the metric average of two compact sets in R, which has linear complexity in the total number of intervals, is presented in [2]. This algorithm can be applied to the reconstruction of $2D$ shapes from their $1D$ cross-sections. The computation of the metric average of compact sets in R^2, required for the reconstruction of $3D$ objects from their $2D$ cross-sections, is much more complicated. As a first attempt, [7] presents an algorithm for the computation of the metric average of two intersecting convex polygons having linear complexity in the number of vertices of the two polygons. This algorithm is generalized for the case of two intersecting regular polygons, but with quadratic computation time [8].

The authors stipulate that the lack of a general approximation result in the case of the Bernstein operators in contrast to the cases of the Schoenberg operators and spline subdivision operators [4] is due to the global nature of the Bernstein operators. In the Bernstein operators the approximation at a point depends on values of the approximated function over all the interval of approximation, while in the two other operators it depends on a finite number of samples of approximated function near the point. This failure of the adaptation method, based on the metric average, lead the authors to extend the metric average to a new set-operation acting on a finite sequence of compact sets. With this operation, most known approximation methods for real-valued functions, are adapted to set-valued functions successfully [6]. Yet at this stage the results are mainly theoretical.

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