Triangulating Planar Point Sets and Delaunay Triangulations

Dan Halperin, Tel Aviv University

Overview

- preliminaries
- legal triangulations
- Delaunay graphs
- algorithms

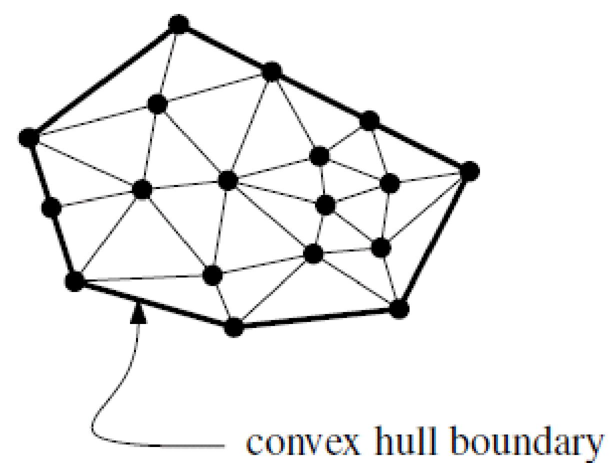
Credits

- the presentation is based on Sections 9.1 and 9.2 of the book by de Berg et al [CGAA]
- the original figures and pseudocode, as well as Ch. 9 in full are available from the book site: www.cs.uu.nl/geobook/

Triangulating planar point sets

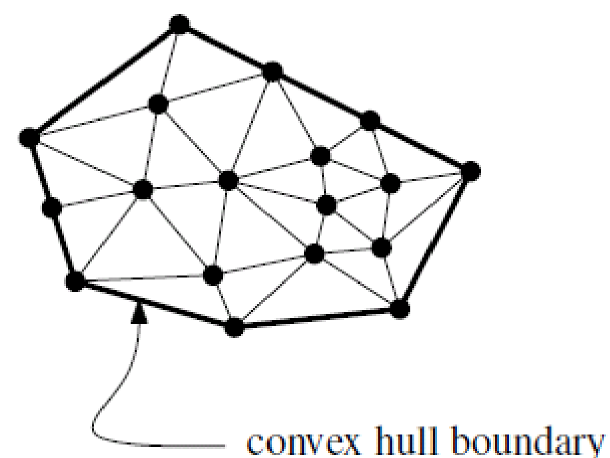
Triangulating point sets

- input: a set P of n points in the pane
- triangulation: by adding a maximal set of non-crossing segments connecting pairs of input points
- why is this a triangulation?



Triangulating planar point sets, size

- if k points of P are on the convex hull of P (possibly in non-general position) then any triangulation of P has
  - $m := 2n - 2 - k$ triangles, and
  - $3n - 3 - k$ edges

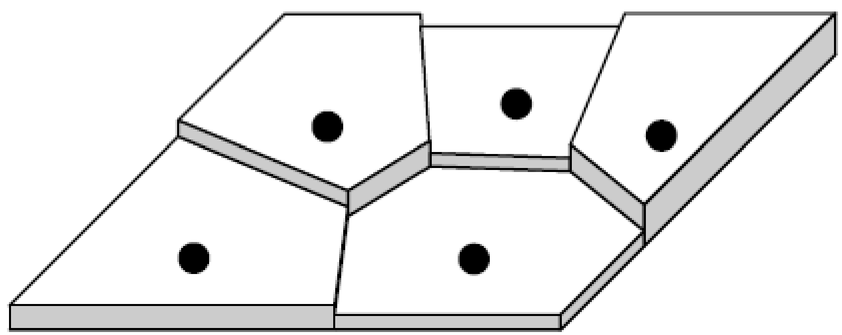


Motivation : Terrains

- Given a set P of data points in the plane
- the height $f(p)$ is determined for each p in P
- how can we naturally approximate the height of points not in P?

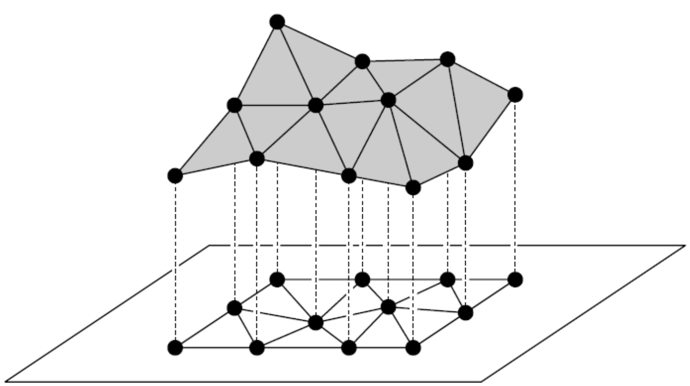
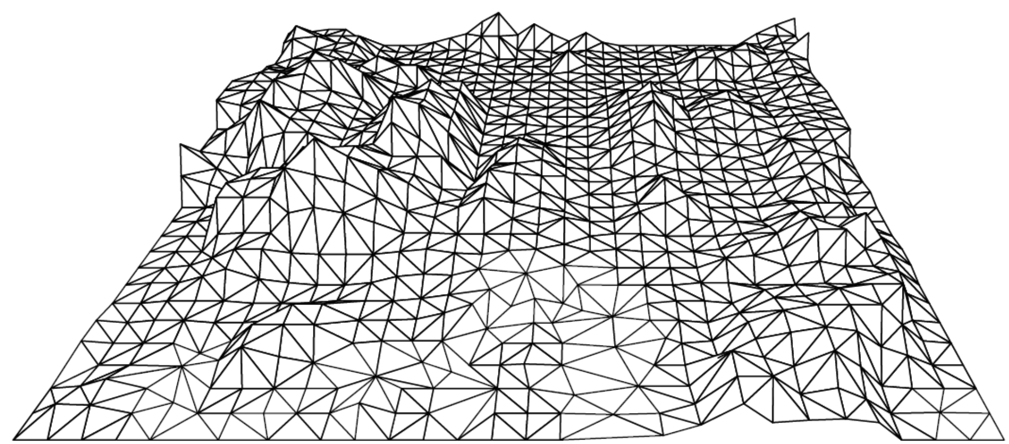
Take I

- let the height of each point not in P be the same as the height as of its closest point in P
- does not look natural



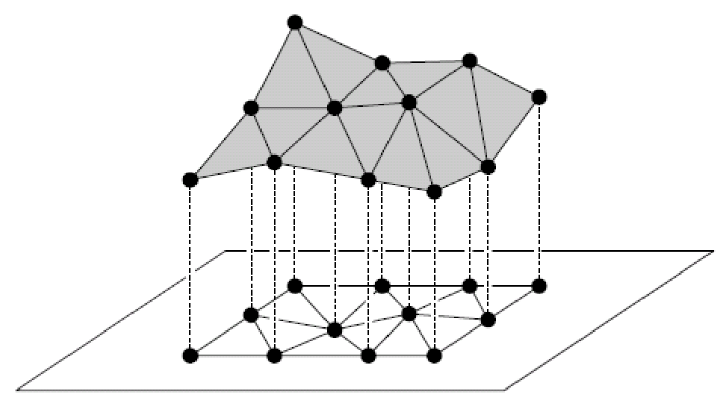
Take II

- determine a triangulation of the points P in the plane, and raise each point to $f(p)$

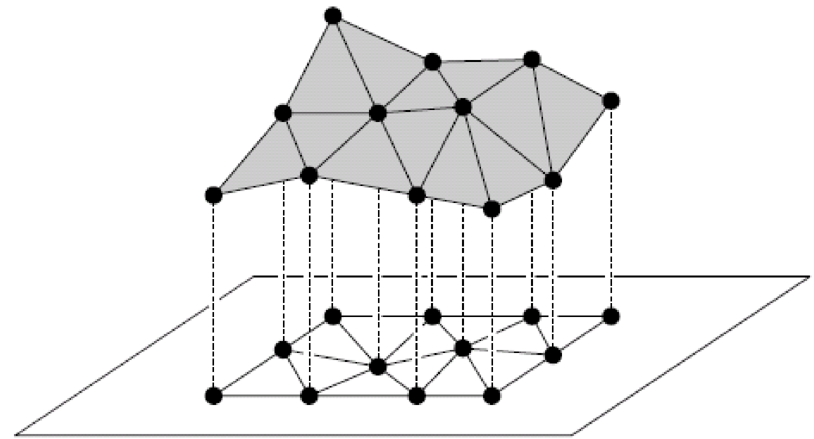
Motivation

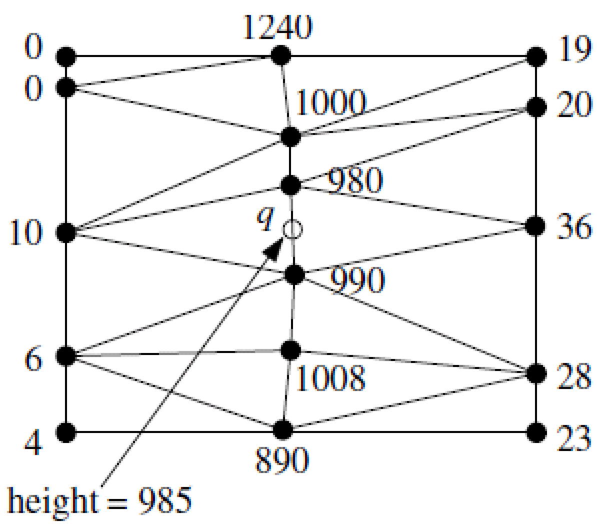
- triangulations are at the heart of central modeling tools in science and engineering
- FEM, meshing
- polyhedral terrains

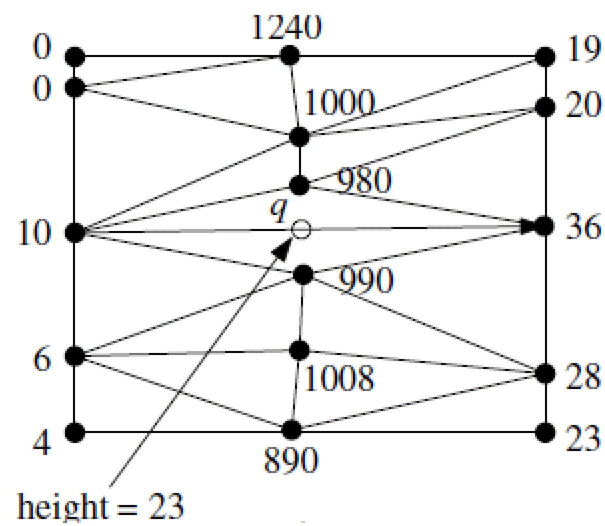


Polyhedral terrains

- some triangulations are better than other







Max min angle

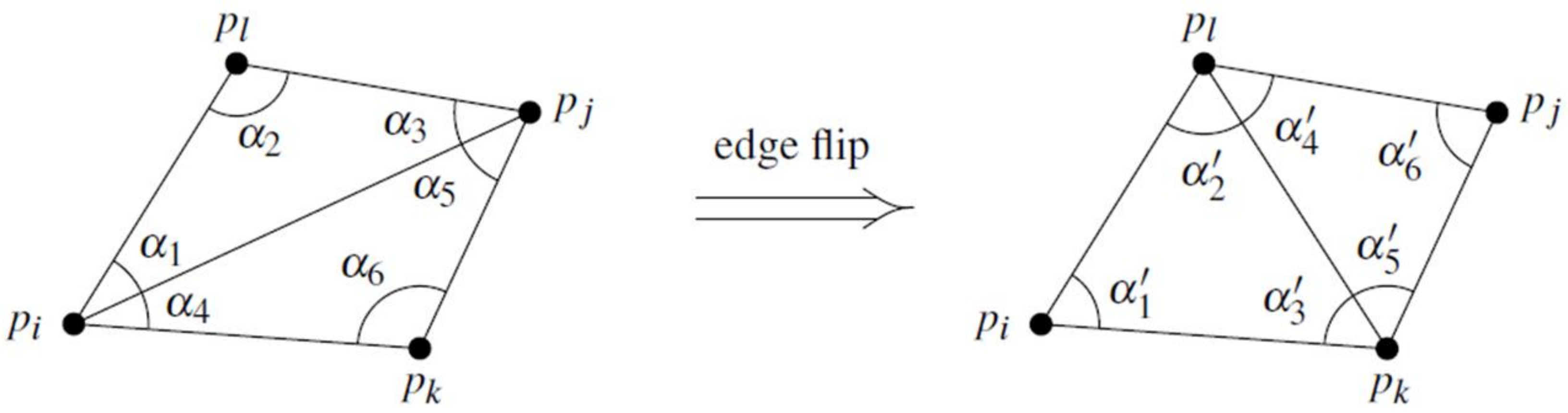
- in what follows, we wish to maximize the minimal angle in the triangulation
- $T(P)$: triangulation of the set of points P
- $A(T)$: the angle vector of the triangulation sorted in increasing order, $(\alpha_1, \alpha_2, \ldots, \alpha_{3m})$
- $A(T)$ is (lexicographically) larger than $A(T')$ if there exists an index $i \in [1,3m]$ such that
  - $\alpha_j = \alpha'_j$ for all $j < i$, and
  - $\alpha_i > \alpha'_i$

Angle-optimal triangulation

- Triangulation T is angle optimal if $A(T) \geq A(T')$ for all triangulations T' of P

Edge flip

- e is an edge of T bounding two triangles that form a convex quadrilateral, then one can perform an edge flip

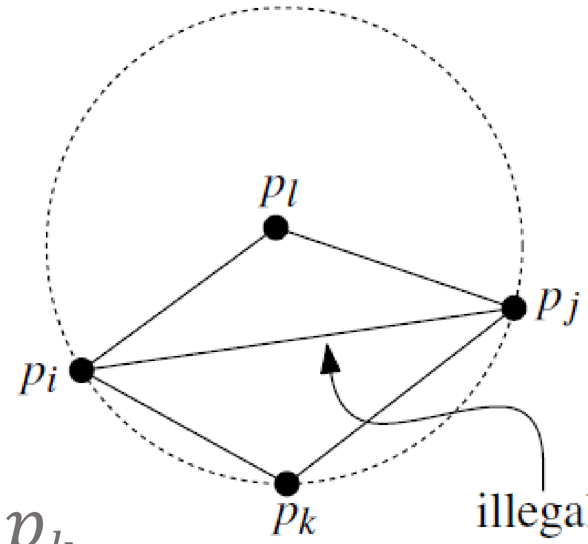


- $p_i p_j$ is an illegal edge if $\min_{1 \le i \le 6} \alpha_i < \min_{1 \le i \le 6} \alpha'_i$

Observation

- let T be a triangulation with illegal edge e
- let T' be obtained from T by flipping e
- then $A(T') > A(T)$

Lemma *illegal*

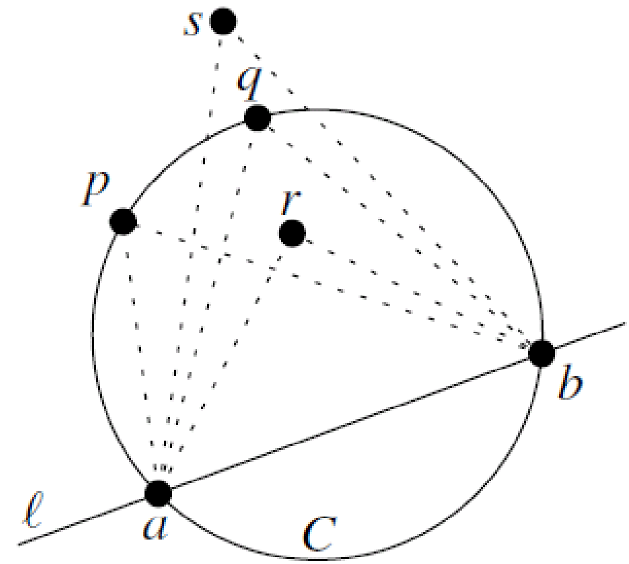


- Let p_ip_j be an edge incident to triangles $p_ip_jp_k$ and $p_ip_jp_l$
- Let C be a circle through p_i, p_j, and p_k
- The edge p_ip_j is illegal iff the point p_l lies inside C
- If p_i, pj, pk, and p_l form a convex quadrilateral and do not lie on a common circle, then exactly one of p_ip_j and p_kp_l is an illegal edge
- the situation is symmetric in p_k and p_l

Theorem (Thales, variant)

- let C be a circle and L a line intersecting C in points a and b
- the points p, q, r, and s all lie on the same side of L
- suppose p and q lie on C, r lies inside C and s lies outside C
- then

  $\measuredangle arb > \measuredangle apb = \measuredangle aqb > \measuredangle asb$

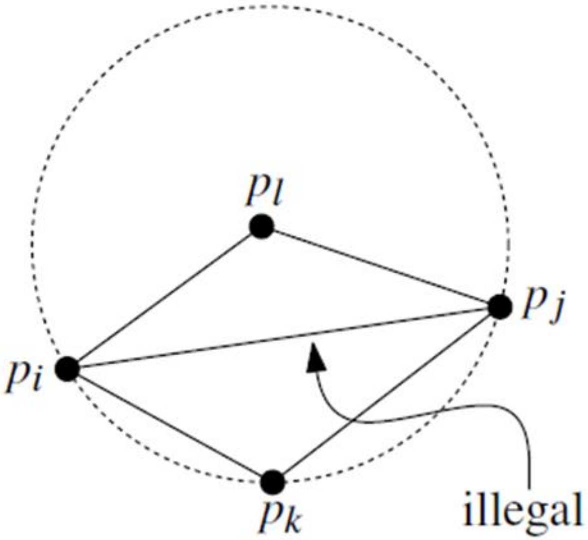


Proof of Lemma *illegal*

- Reminder: the edge $p_i p_j$ is illegal iff the point p_l lies inside C

- using Thales theorem
- for example:

$\measuredangle p_i p_l p_k > \measuredangle p_i p_j p_k$



Point inside a circle

- the lemma gives us an easy way to test the validity of an edge
- there is a simple and elegant predicate to test if a point lies inside a circle–we will see it later

Legal triangulations

- a legal triangulation is a triangulation that does not contain any illegal edge
- any angle-optimal triangulation is legal

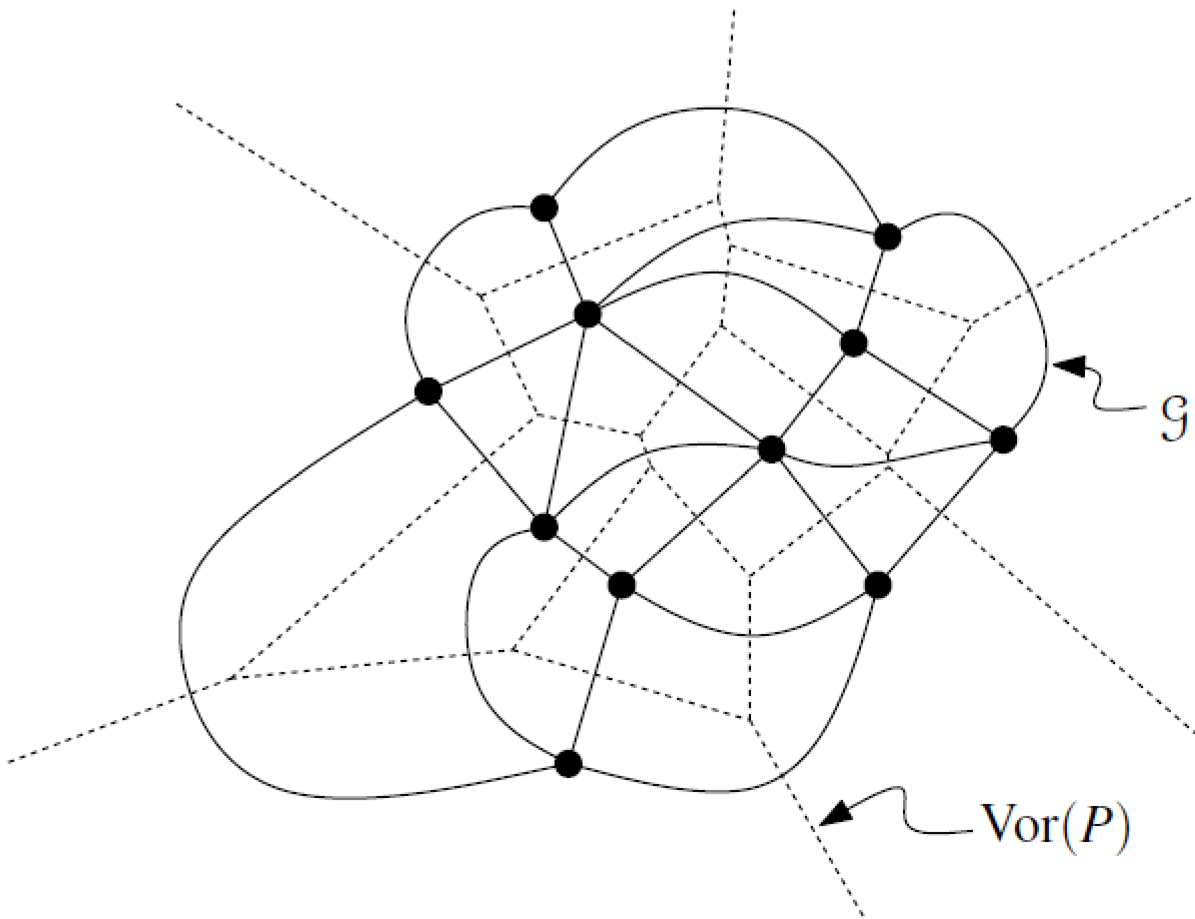
Algorithm LEGALTRIANGULATION($\mathcal{T}$)
Input. Some triangulation $\mathcal{T}$ of a point set P.
Output. A legal triangulation of P.
1. **while** $\mathcal{T}$ contains an illegal edge $\overline{p_i p_j}$
2. **do** (* Flip $\overline{p_i p_j}$ *)
3. Let $p_i p_j p_k$ and $p_i p_j p_l$ be the two triangles adjacent to $\overline{p_i p_j}$.
4. Remove $\overline{p_i p_j}$ from $\mathcal{T}$, and add $\overline{p_k p_l}$ instead.
5. **return** $\mathcal{T}$

Legal triangulation by edge flips

- why is the loop finite?

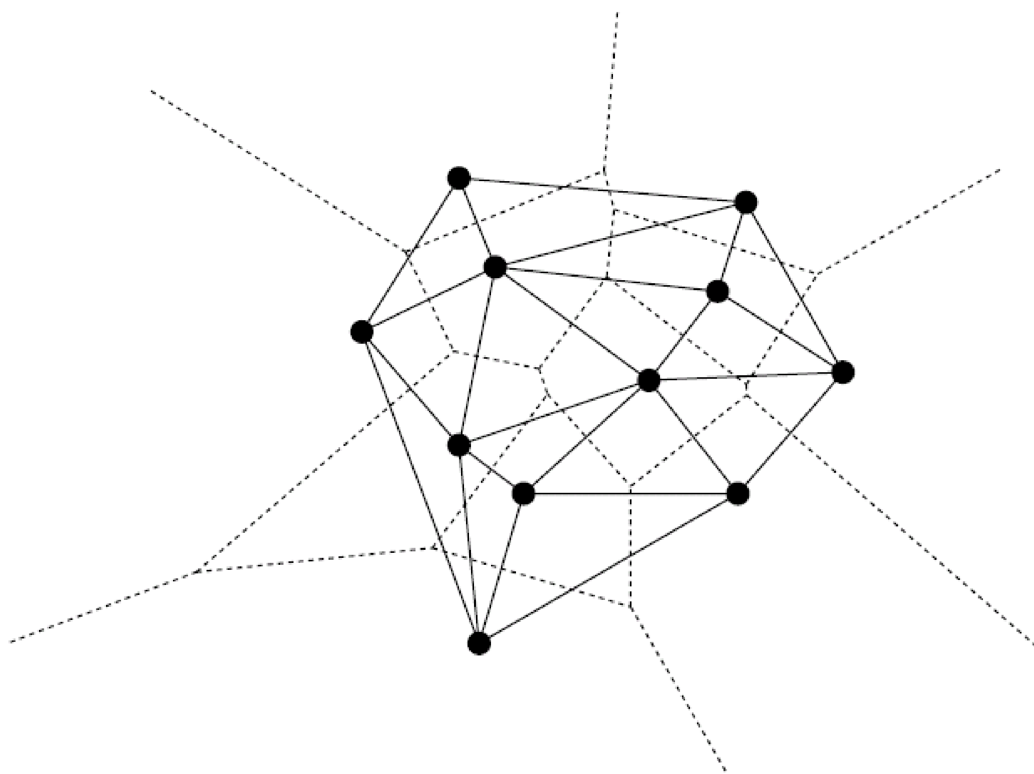
The Delaunay Graph

The (graph) dual of the Voronoi diagram



The Delaunay Graph

- The straight edge dual of the Voronoi diagram

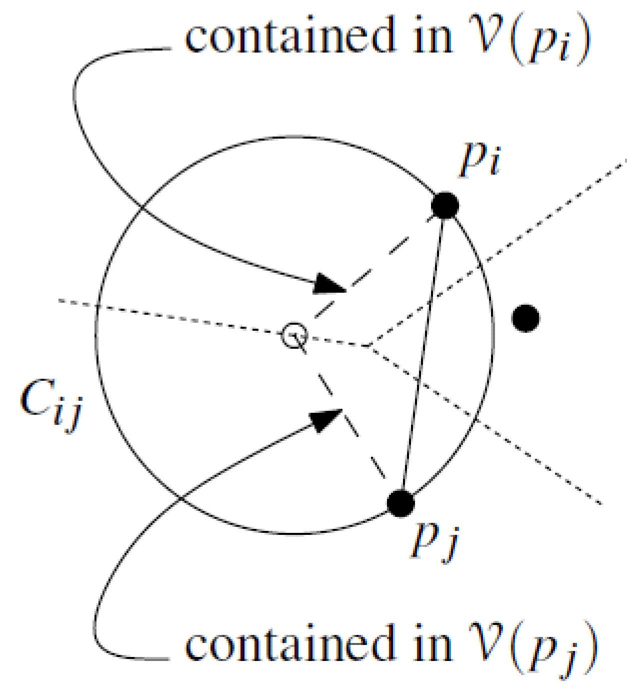


Theorem

- The Delaunay graph of a planar point set is a plane graph.

C_{ij}: an empty circle through p_i, p_j
c_{ij}: the center of C_{ij}, lying on the Voronoi edge between $V(pi)$ and $V(p_j)$
t_{ij}: the triangle c_{ij}, pi, pj

t_{kl}: the same for the sites p_k, pl

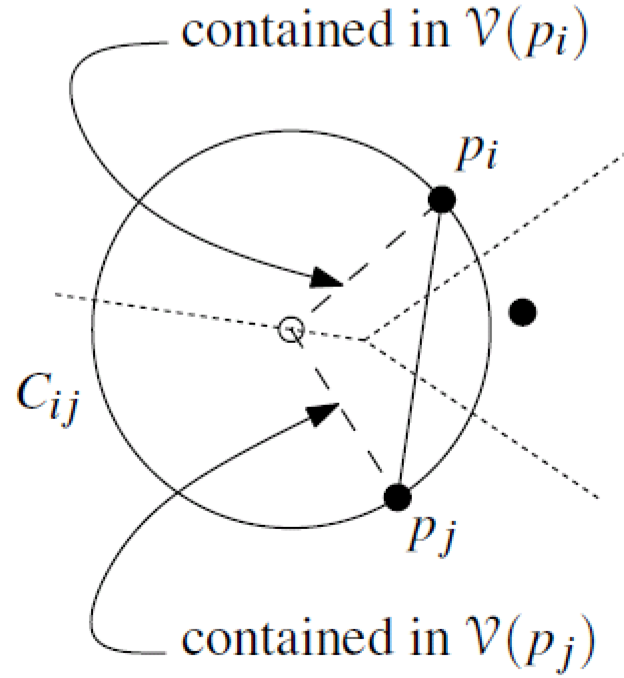


Proof

- suppose for a contradiction that

$p_i p_j$ and $p_k p_l$ intersect

- notice that p_k and p_l must lie outside C_{ij}, and therefore outside t_{ij}
- this implies that $p_k p_l$ must intersect one of the edges of t_{ij} incident to c_{ij}
- similarly, $p_i p_j$ must intersect one of the edges of t_{lk} incident to c_{ij}
- it follows that one of the edges of t_{ij} incident to c_{ij} must intersect one of the edges of t_{kl} incident to c_{kl}
- contradiction: each edge incident to c_{ij} or c_{kl} must lie in a unique Voronoi cell



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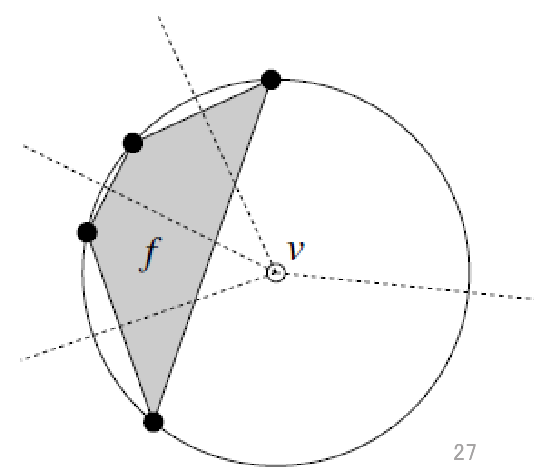
The Delaunay Triangulation

- If we assume general position
  - no three points are collinear, and
  - no four points are cocircular
- then the Delaunay graph is a triangulation.

- (Otherwise, the Delaunay graph can be easily triangulated.)

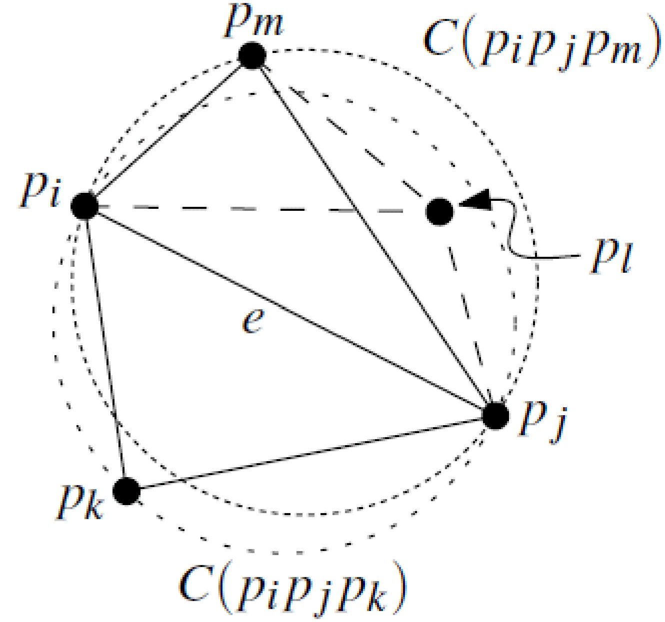


Theorem VD2DG

- We rephrase a theorem on Voronoi diagrams in terms of their straight edge dual:
  - three points p_i, p_j, p_k in P are vertices of the same face of the Delaunay graph iff the circle through p_i, p_j, p_k contains no point of P in its interior
  - the points p_i, p_j in P form an edge of the Delaunay graph iff there is a closed disc that contains p_i, p_j on its boundary and does not contain any other point of P
- Corollary: A triangulation T of a point set P is a Delaunay triangulation of P iff the circumcircle of any triangle in T does not contain a point of P in its interior

Theorem (almost the punch line)

- Thm: A triangulation T of a point set P is legal iff T is a Delaunay triangulation of P
- Pf: Assume the contrary, and let $(t, pl),\ t\ \in T$, and p_l inside circumcircle(t), such that $\measuredangle p_i p_j p_l$ is the largest



The punch line(s)

- Theorem: Any angle-optimal triangulation of a point set P is a Delaunay triangulation of P.

- Any Delaunay triangulation of P maximizes the minimum angle over all triangulations of P.

- We say *a* Deluanay triangulation since we do not assume general position. We also rely on the fact that in any triangulation of cocircular points the minimal angle is the same.

Algorithms

- construct the Voronoi diagram and dualize (highly inefficient in terms of algebraic operations)

- randomized incremental construction [GKS]

- we will see another algorithm, based on CH computation, later

THE END