Exact Construction of Minimum-Width Annulus of Disks in the Plane

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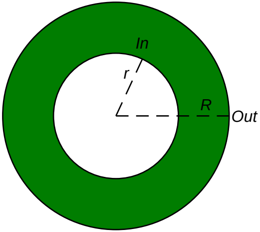
25^{th} European Workshop on Computational Geometry
Brussels, Belgium, March 2008

Minimum-Width Annulus

- An annulus is the bounded area between two concentric circles
- The width of an annulus is the difference between its radii R and r
- **Goal:** given a set of objects (points, segments, disks etc.) find an annulus of minimum width containing the objects (MWA)



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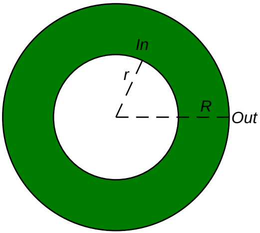
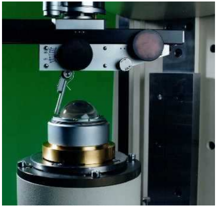


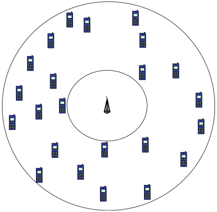
Applications

Tolerancing Metrology — MWA can be used to calculate the roundness of a manufactured object (how much the object is round)



www.npl.co.uk/server.php

Facility Location — Location of facilities with both desirable and obnoxious properties



cgm.cs.mcgill.ca/˜athens/cs507/Projects/2004/Emory-Merryman

Related Work

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- Linear and $O(n \log n)$ algorithms for special cases with more information: Swanson et al. ’93, Garcia-Lopez et al. ’98, Devillers and Ramos ’02
- Chan ’06 presented a linear $(1 + \varepsilon)$-factor approximation algorithm based on coresets

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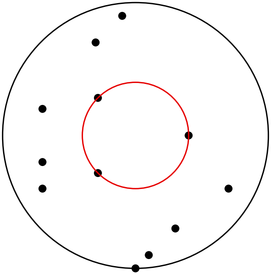
Inner circle touches 3 points — center is a nearest Voronoi vertex



The Connection to Voronoi Diagrams

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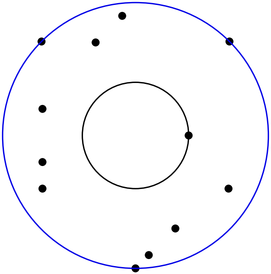
The center of the MWA is a vertex of the overlay of the nearest and furthest Voronoi diagrams

3 cases:

Outer circle touches 3 points — center is a farthest Voronoi vertex



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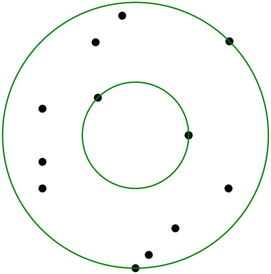
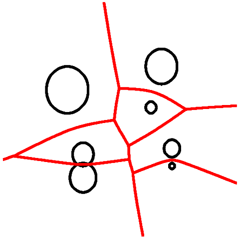
3 cases:

Both inner and outer circles touches $\geq$ 2 points — center is an intersection point between the diagrams (on edges of both diagrams)



MWA of Disks in the Plane

Nearest Voronoi is replaced by the Apollonius diagram



$$\delta(x, d_i) = ||x - c_i|| - r_i$$

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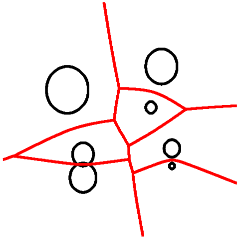
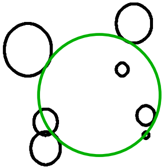
Farthest Apollonius diagram is not good





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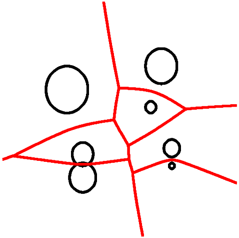

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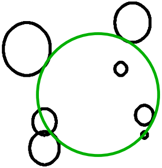
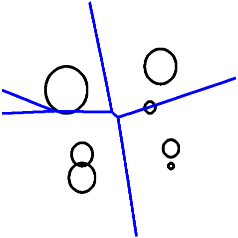


We need to consider the farthest point of the disk from a point

Farthest-Point Farthest-Site VD replaces the farthest VD



$\delta(x, d_i) = ||x - c_i|| + r_i$

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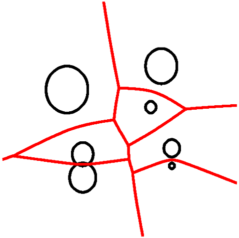

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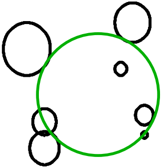

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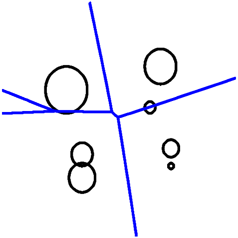
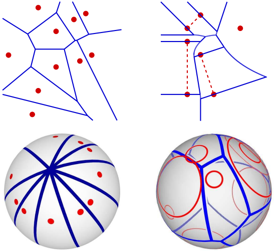


$\delta(x, d_i) = ||x - c_i|| + r_i$

Both diagrams have hyperbolic bisectors

Exact Implementation using CGAL

- VD are computed using a randomized Voronoi framework based on CGAL `Envelope_3` and `Arrangement_2` packages
- Both diagrams implementations are based on the ability to compute arrangements of algebraic plane curves (Eigenwillig and Kerber ’08)



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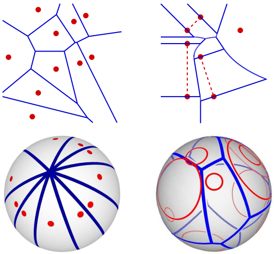
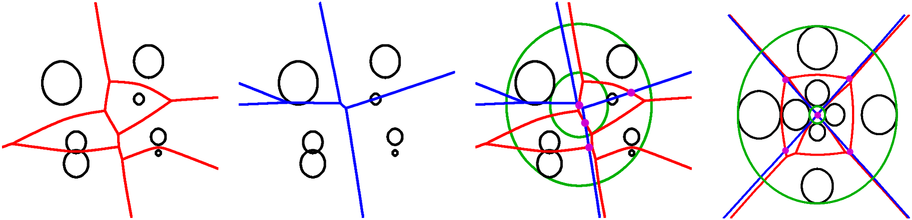


- The implementation is robust — all (degenerate) inputs are handled correctly with exact number types
- We use the overlay function of CGAL to overlay the two diagrams
- Total exp. time: $O(n \log^2 n + k \log n)$

Results



No. Disks	Time (secs)	V	E	F
50	10.741	126	213	88
100	26.994	238	395	158
200	62.968	416	659	244
500	185.244	775	1174	400
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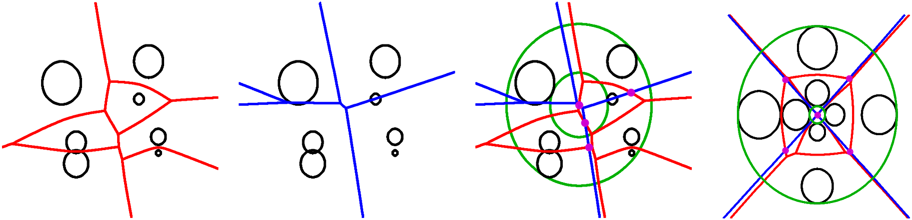

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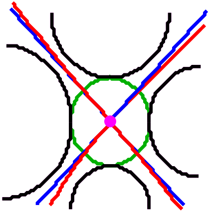

THE END

Further Reading I

- Mark de Berg, Prosenjit Bose, David Bremner, Suneeta Ramaswami, and Gordon T. Wilfong.
  Computing constrained minimum-width annuli of point sets.
  Computer-Aided Design, 30(4):267–275, 1998.

- Theodore J. Rivlin.
  Approximating by circles.
  Computing, 21:93–104, 1979.

- Michiel H. M. Smid and Ravi Janardan.
  On the width and roundness of a set of points in the plane.
  International Journal of Computational Geometry and Applications, 9(1):97–108, 1999.

Further Reading II

- Hiroyuki Ebara, Noriyuki Fukuyama, Hideo Nakano, and Yoshiro Nakanishi.
  Roundness algorithms using the Voronoi diagrams.
  In *Proceedings of 16th The Canadian Conference on Computational Geometry*, volume 41, 1989.
- Utpal Roy and Xuzeng Zhang.
  Establishment of a pair of concentric circles with the minimum radial separation for assessing roundness error.
  Computer-Aided Design, 24(3):161–168, 1992.
- Van-Ban Le and Der-Tsai Lee.
  Out-of-roundness problem revisited.
  IEEE Transactions on Pattern Analysis and Machine Intelligence, 13(3):217–223, 1991.

Further Reading III

Arno Eigenwillig and Michael Kerber.
Exact and efficient 2D-arrangements of arbitrary algebraic curves.
In *Proceedings of 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 122–131, Philadelphia, PA, USA, 2008. Society for Industrial and Applied Mathematics (SIAM).

Timothy Moon-Yew Chan.
Faster core-set constructions and data-stream algorithms in fixed dimensions.
Computational Geometry: Theory and Applications, 35(1-2):20–35, 2006.