Constructing Two-Dimensional Voronoi Diagrams via Divide-and-Conquer of Envelopes in Space

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Tel Aviv University, Israel

Thesis Exam

Tel Aviv University, June 2009

Contribution of the Thesis

- General framework for computing two-dimensional Voronoi diagrams
  - First exact implementation of several types of diagrams:
    - ★ Möbius diagrams
    - ★ Anisotropic diagrams
    - ★ Farthest-site Voronoi diagrams
    - ★ and more

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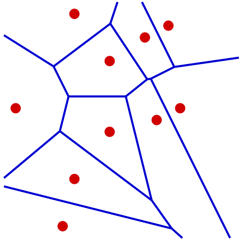
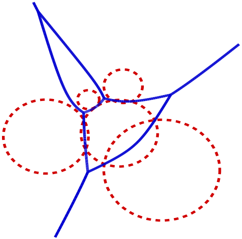
- Computing minimum-width annulus of a set of disks in the plane: Theory and practice
- Contributions to the Computational Geometry Algorithms Library

Available Diagrams

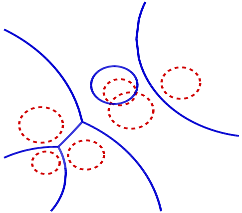
Nearest-Site Voronoi Diagrams



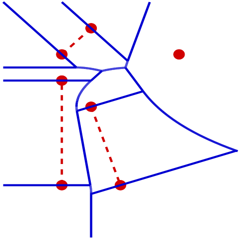
Standard Voronoi diagrams and power diagrams



Apollonius (additively-weighted Voronoi) diagrams



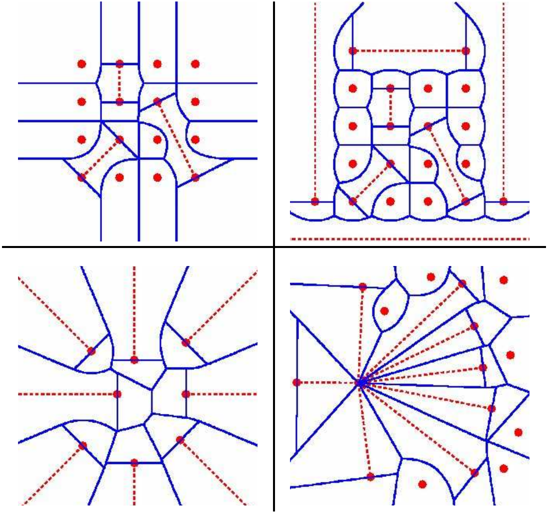
Möbius and anisotropic diagrams



Voronoi diagram of linear objects

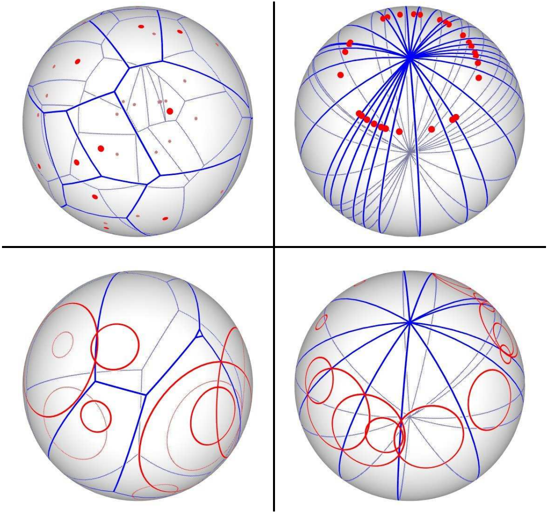
Available Diagrams

More Diagrams of Linear Objects



Available Diagrams

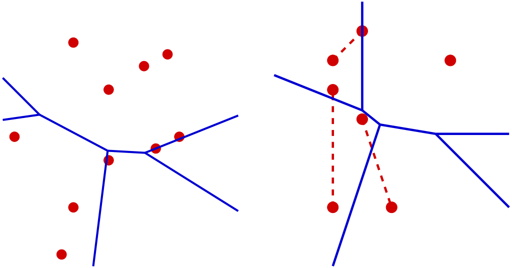
On the Sphere



Available Diagrams

Others

- Farthest-site Voronoi Diagrams:



- Two-site triangle-area distance function Voronoi diagram

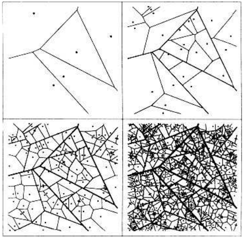
Outline

1. Overview
2. Introduction
3. The Algorithm
4. Implementation Details
5. Application: Minimum-Width Annulus
6. Future Work

Voronoi Diagrams

- Given n objects (Voronoi sites) in some space (e.g., $\mathbb{R}^d$, $\mathbb{S}^d$) and a distance function ρ
- The Voronoi Diagram subdivides the space into cells
- Each cell consists of points that are closer to one particular site than to any other site

- Variants include different:
  - Classes of sites
  - Embedding spaces
  - Distance functions (e.g., farthest-site Voronoi diagrams)



Fractals from Voronoi diagrams
http://www.righto.com/fractals/vor.html

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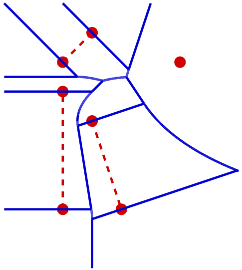
Voronoi diagram of segments

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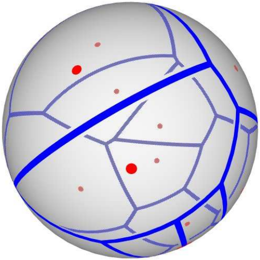

Voronoi diagram on the sphere

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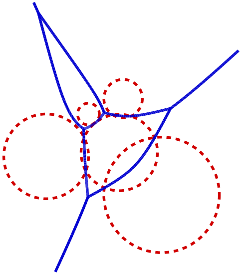

Apollonius diagram

Some of the Related Work

About 1 in 16 papers in computational geometry deals with Voronoi diagrams [Aurenhammer & Klein, 2000]

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Theoretical:

- Edelsbrunner and Seidel [1986] observed the connection between Voronoi diagrams and envelopes
- Randomized incremental construction [Guibas et al., 1992]
- Abstract Voronoi diagrams [Klein, 1989, Klein et al., 1993]

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Practical but non-exact:

- VRONI code for two-dimensional Voronoi diagrams of points and line segments (and circular arcs) [Held, 2001]
- Use of Graphics Processing Unit (GPU) [Hoff III et al., 1999, Nielsen, 2008]

Some of the Related Work

Cont.

Practical and exact:

- New D&C [Aichholzer et al., 2009] computes bounded Euclidean VD in the plane
- CGAL Delaunay graphs for computing standard Voronoi diagrams, Apollonius diagrams, and segment Voronoi diagrams [Boissonnat et al., 2002, Emiris & Karavelas, 2006, Karavelas, 2004].
- Segment Voronoi diagrams [Burnikel et al., 1994] and Randomized incremental construction for abstract VD [Seel, 1994] in LEDA

Lower Envelopes

and Voronoi Diagrams

Definition

Given a set of bivariate functions $S = \{s_1, \ldots, s_n\}$, their lower envelope is defined to be their pointwise minimum:

$$\Psi(x,y) = \min_{1 \le i \le n} s_i(x,y)$$

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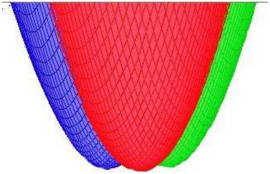
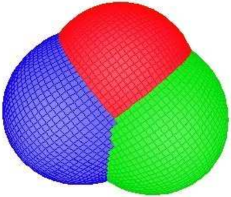
Corollary

Voronoi diagrams are the minimization diagrams of the distance functions from each site [Edelsbrunner & Seidel, 1986]



Distance functions are paraboloids

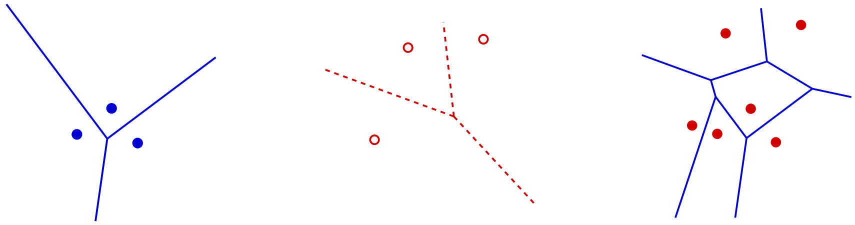


Looking from bottom gives us the Voronoi diagram

The Divide-and-Conquer Algorithm

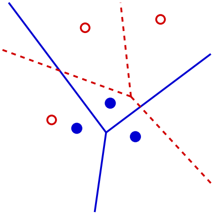
Let S be a set of n sites

1. Partition S into two disjoint subsets S_1 and S_2 of equal size
2. Construct $\mathrm{Vor}_\rho(S_1)$ and $\mathrm{Vor}_\rho(S_2)$ recursively
3. Merge the two Voronoi diagrams to obtain $\mathrm{Vor}_\rho(S)$



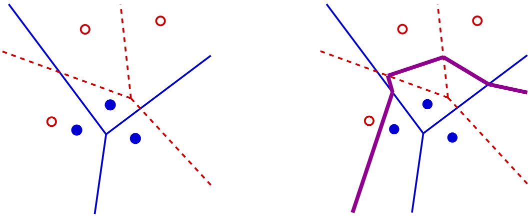
The Merging Step

1. Overlay $\mathrm{Vor}_\rho(S_1)$ and $\mathrm{Vor}_\rho(S_2)$



The Merging Step

1. Overlay $\mathrm{Vor}_\rho(S_1)$ and $\mathrm{Vor}_\rho(S_2)$
2. Partition each face to points closer to a site in S_1 and points closer to a site in S_2
3. Label feature of the refined overlay with the sites nearest to it



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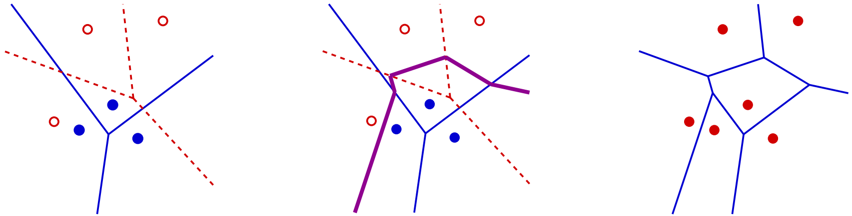
4. Remove redundant features



Complexity

- Theoretical worst-case time complexity is quadratic even for diagrams of linear complexity

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- Using randomization we get a better complexity

Theorem (Sharir)

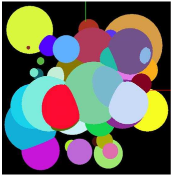
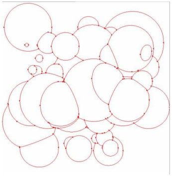
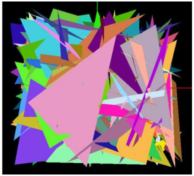
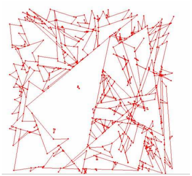
For a type of two-dimensional Voronoi diagrams of complexity $F(n)$, if we randomly split the sites into two subsets then the expected complexity of the overlay of the Voronoi diagrams is $O(F(n))$.

Corollary

For a type of two-dimensional Voronoi diagrams of linear complexity the divide-and-conquer envelope algorithm computes it in expected $O(n \log^2 n)$ time. For a type of two-dimensional Voronoi diagrams of $F(n) \in \Omega(n^{1+\varepsilon})$ complexity, the expected running time is $O(F(n) \log n)$.

Envelopes and Arrangements in CGAL

- `Envelope_3` package in CGAL constructs lower and upper envelopes of general surfaces [Meyerovitch, 2006]
- Based on the `Arrangement_on_surface_2` (previously `Arrangement_2`) package
- Robust and exact
  - All inputs are handled correctly (including degenerate inputs)
  - Exact number types are used to achieve exact results
- Efficient
- Part of the CGAL basic library

Implementation and Interface

- Reduced and simplified interface for diagrams with one-dimensional bisectors
- Computing diagrams the curves of which are currently supported by the arrangement package is made easy
- The framework supports types of diagrams that most frameworks do not support:
  - Quadratic-size diagrams, e.g., Möbius diagrams and triangle-area distance-function Voronoi diagrams
  - Non-connected bisectors, e.g., anisotropic Voronoi diagrams
  - Two-dimensional bisectors

Other Advantages

The diagrams are represented as CGAL arrangements

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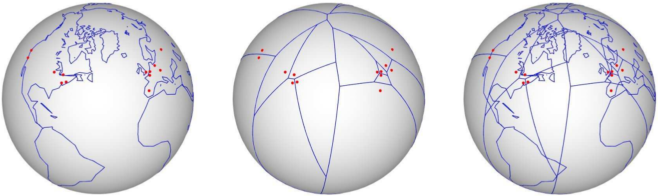
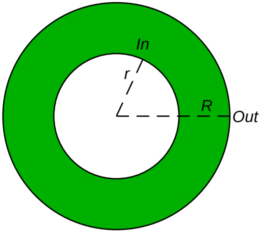
- The vertices, edges, and faces of the diagrams can easily be traversed while obtaining coordinates to any desired precision
- Point-location functionality
- Inserting and removing curves
- Overlay between diagrams, which is used, for example, for computing minimum-width annulus and for representing the local zones of two competing telecommunication operators [Baccelli et al., 2000]
- etc.

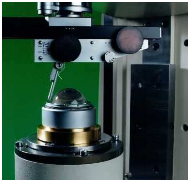


Overlaying an arrangement and a Voronoi diagram on the sphere

Application: Minimum-Width Annulus

- **Goal:** Given a set of disks in the plane, find an annulus of minimum width containing the disks
- Minimum-width annulus (MWA) has applications in tolerancing metrology and facility location
- We extended a known algorithm for computing a minimum-width annulus of points [Ebara et al., 1989] to disks





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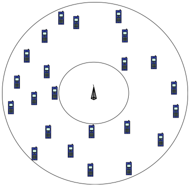
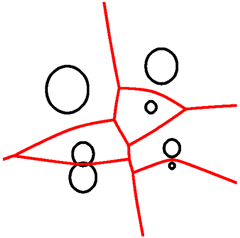


cgm.cs.mcgill.ca/˜athens/cs507/Projects/2004/Emory-Merryman

MWA of Disks in the Plane

Nearest Voronoi is replaced by the Apollonius diagram

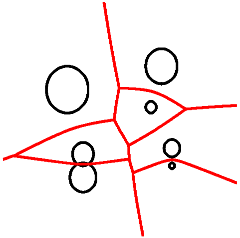


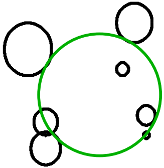
$$\delta(x, d_i) = ||x - c_i|| - r_i$$

MWA of Disks in the Plane

Nearest Voronoi is replaced by the Apollonius diagram

Farthest Apollonius diagram is not good in this case





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We need to consider the farthest point of the disk from a point

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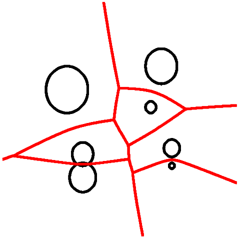

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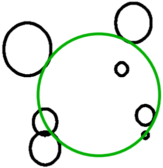

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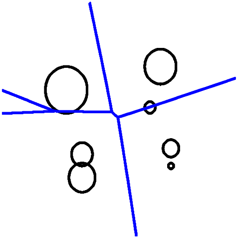
Farthest-Point Farthest-Site VD replaces the farthest VD



$$\delta(x, d_i) = ||x - c_i|| + r_i$$

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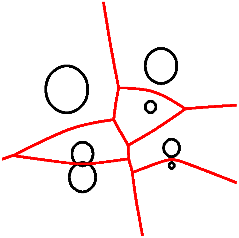
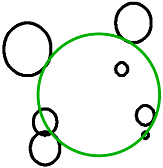
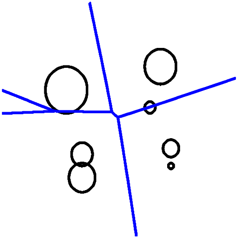






$\delta(x, d_i) = ||x - c_i|| - r_i$

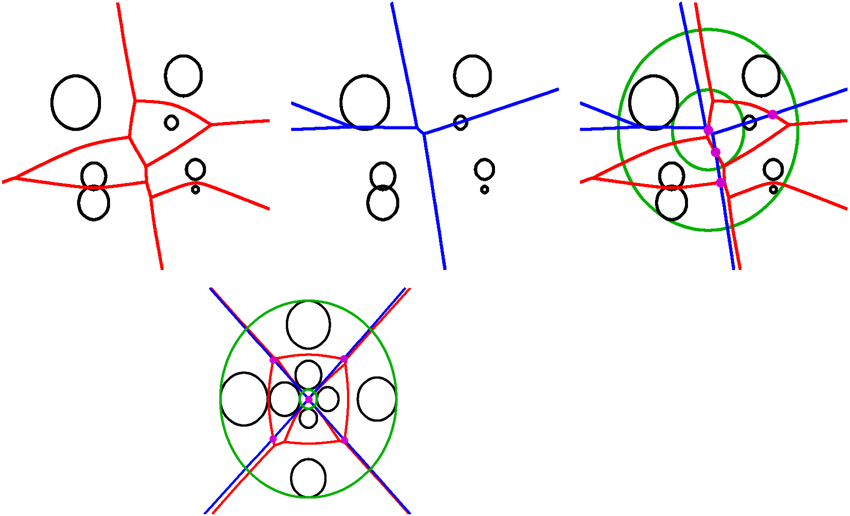
We need to consider the farthest point of the disk from a point

$\delta(x, d_i) = ||x - c_i|| + r_i$

Farthest-point farthest-site is a farthest-site Apollonius with negative radii and was easily produced using our framework

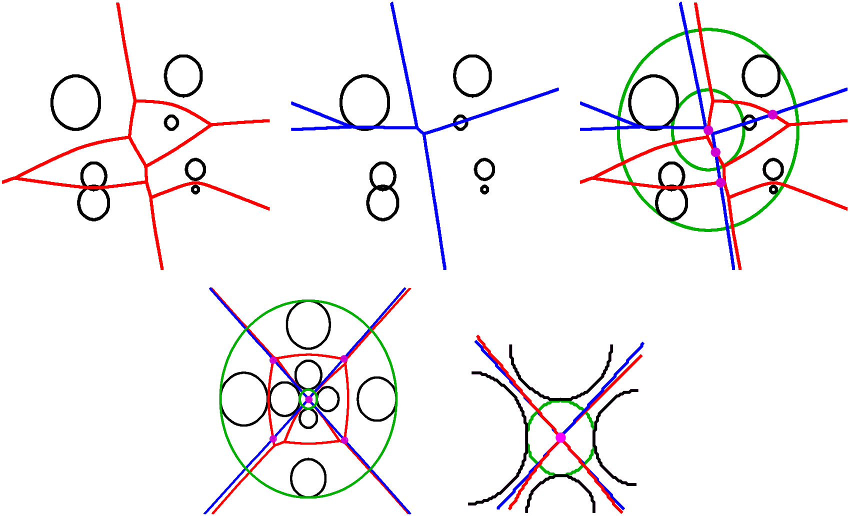
MWA of Disks in the Plane

Cont.



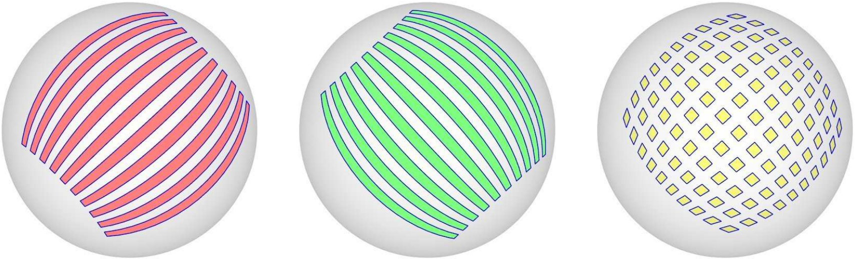
MWA of Disks in the Plane

Cont.



Contributions to CGAL

- Contribution in the development of the `Arrangement_on_surface_2` package
- Extending the `Envelope_3` package to support envelopes embedded on two-dimensional surfaces
- Enhancing existing traits classes for the arrangement package
  - Extending the circular-arcs traits to support unbounded lines (for the Möbius diagram)
  - Contribution to the traits class that supports algebraic plane curves



Boolean set operation on the sphere

Future Work

- Enrich the variety of Voronoi diagrams computed with the framework
  - Voronoi diagrams of circular arcs
  - Voronoi diagrams of weighted segments
  - Voronoi diagrams under the Manhattan metric
  - Hausdorff Voronoi diagrams
  - Diagrams embedded on other two-dimensional surfaces
  - and more

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- Improve the performance of the code in practice
  - Avoid overlaying the entire arrangements in the merge step
- Minimum-width annuli of other types of objects

Literature I

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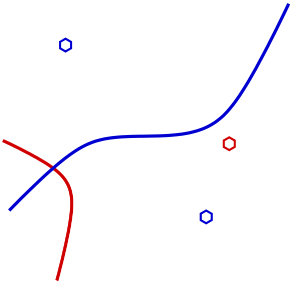
Avoiding Filter Failures: The 3-Bisector Optimization

Observation

Given three Voronoi sites and their pairwise bisectors, if two of the bisectors intersect then the third bisector passes through their intersection point(s)



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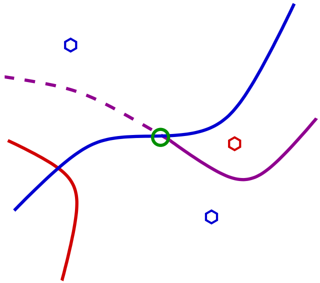

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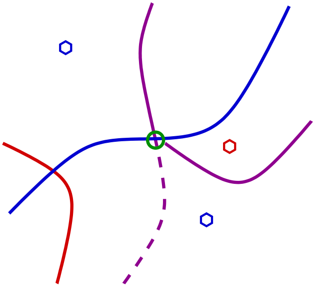
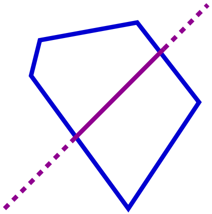


Conclusion

We can prevent filter-failures caused by inserting a curve into an arrangement when the curve passes exactly through an existing vertex of the arrangement

Simplified Zone Algorithm

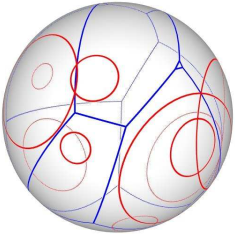
- CGAL uses the zone algorithm to partition each face with the bisector
- The zone algorithm does redundant work when it comes to the standard Voronoi diagram (all the faces are convex)
- *Using a simplified version of the zone algorithm gives us an additional performance improvement*



Power Diagrams on the Sphere

and Applications

- Applications of diagrams on the sphere:
  - Properties of the spherical Voronoi diagram are used to prove the *the thirteen spheres* theorem [Anstreicher, 2004]
  - Power diagram on the sphere are used to determine whether a query point is in the union of circles on the sphere
  - Find connected components of sets of circles on the sphere



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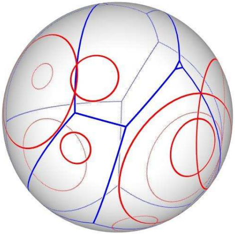

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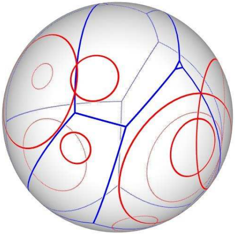

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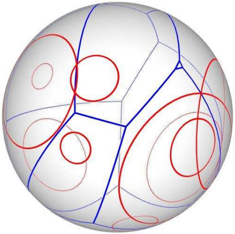
- Bisectors are great circles
- Larger circles have greater influence locally, but less influence on the other side of the sphere
- There can be empty cells



Comments on Triangle-Area Voronoi Diagram

- The triangle area distance function between a point $x \in \mathbb{R}^2$ to a Voronoi site $\{p, q\}$ is defined by the area of $\triangle xpq$
- The bisector of two sites $\{p_1, q_1\}$ and $\{p_2, q_2\}$ consists of two intersecting rational lines in the general case

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- If the two segments are parallel, then the bisector is a pair of parallel lines that become the same line if the lengths of the two segments is equal
- If the two segments are collinear then the bisector is either one line, or, in case that of equal lengths does not exist.
  - Both sides of the one line are dominated by the larger segment