Supplementary On-line Proofs for *Sampling-Diagram Automata: a Tool for Analyzing Path Quality in Tree Planners*

Oren Nechushtan[1*], Barak Raveh[12*], and Dan Halperin[1]

[1] School of Computer Science, Tel-Aviv University,
{theoren,barak,danha}@post.tau.ac.il *
[2] Dept. of Microbiology and Molecular Genetics, IMRIC, The Hebrew University

Additional Lemmas used for proving Theorem 2 in Section 4

Note that the main lemma used in the proof of Theorem 2 is Lemma 7 below, and the other lemmas are used as auxiliary geometric lemmas. For completeness, we include in this document a figure of the Promenade motion-planning problem, as it appears in the main text.

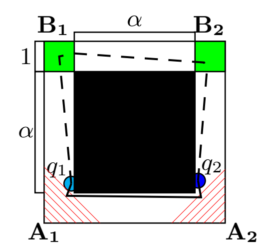

Fig. 1: An illustration of the Promenade motion planning problem, a square obstacle within a square bounding-box. In this example, q_1 and q_2 lie on opposite sides of the promenade. Type-A (solid line) and type-B (dashed line) solution paths between q_1 and q_2 are shown, as defined in Section 4. The ratio between their length, denoted μ, is approximately $\frac{1}{3}$ in this case.

Lemma 3. *Let s denote a non-rejecting state in an ASD $\mathbb{A}$. If $\mathbb{A}$ moves to s after reading a word W with S_{cur} being the swath produced by Bi-RRT, and if*

(i) $R = R(D_\sigma[s])$ *is the intersection of* $\mathcal{C}_{\text{free}}$ *with a half-plane, and*
(ii) $q_1, q_2 \notin R$

* ON and BR contributed equally to this work

then $S_{cur} \cap R = \phi$.

Proof. For any non-rejecting state s, since R is the intersection of a half-plane with $\mathcal{C}$, R forms a "visibility block" in the configuration space. Formally, if $s, t \in \mathcal{C}_{\text{free}} \backslash R$ then the line between s and t does not intersect R due to the convexity of either half-plane. Since $\mathbb{A}$ is an ASD, we know that no sample σ_i ever hit R on its way to the state s. Since also $q_1, q_2 \notin R$, we conclude the swaths produced by Bi-RRT also do not intersect R.

Lemma 4. *Let B_1 denote the top-left square in $\mathcal{C}_{\text{free}}$ and let $G \subset B_1$ denote a smaller homothetic square adjacent to the top-right corner of B_1. Given $1 \leq p \leq \infty$ let L_{top} (resp. L_{bottom}) denote the ℓ_p-bisector of c_1 and the top-(bottom-, resp.) left corner of G. Then, any point r in $\mathcal{C}_{\text{free}}$ to the right of c_1 and above both L_{top} and L_{bottom} is closer to any point in G than to c_1.*

Proof. For any point r to the right of c_1, the ℓ_p-distance from r to a point in G attains its maximum on either the top-left or bottom-left corners of G. Since r is above both bisectors, then c_1 is farther away from r than all points in G.

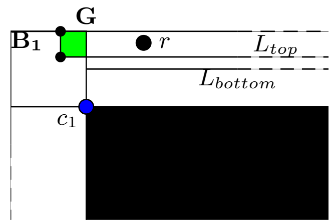

Fig. 2: Let B_1 denote the top-left free square $[0, 1] \times [\alpha + 1, \alpha + 2]$ and let $G \subset B_1$ denote a small homothetic square adjacent to the top-right corner of B_1. Let c_1 and c_2 denote the top-left and top-right corners of the obstacle. Let L_{top} (resp. L_{bottom}) denote the ℓ_1-bisector of c_1 and the top-(bottom-, resp.) left point of G, restricted to the points to the right of c_1 within $\mathcal{C}_{\text{free}}$. Then, for any point r to the right of c_1 and above both L_{top} and L_{bottom}, r is closer to any point in G than to c_1.

See Figure 2 for an illustration of L_{top} and L_{bottom} using the ℓ_1*−norm*.

Notice that generally for $1 < p < \infty$ L_{top} and L_{bottom} may intersect. It is also easy to show that taking G small enough guaranties that the zone in $\mathcal{C}_{\text{free}}$ defined in the previous Lemma by the bisectors and c_1 is not empty. E.g., if G is a $\gamma \times \gamma$ square with $\gamma < 1/2$ then the zone is non-empty for ℓ_1.

Think of S as the S_{cur} induced by $\mathbb{A}_\alpha$.

Lemma 5. *Let (c_1, c_2) denote two adjacent corners of the inner-square in $\mathcal{P}_\alpha$. Let δ denote the open disc around c_2 whose boundary passes through c_1. Let*

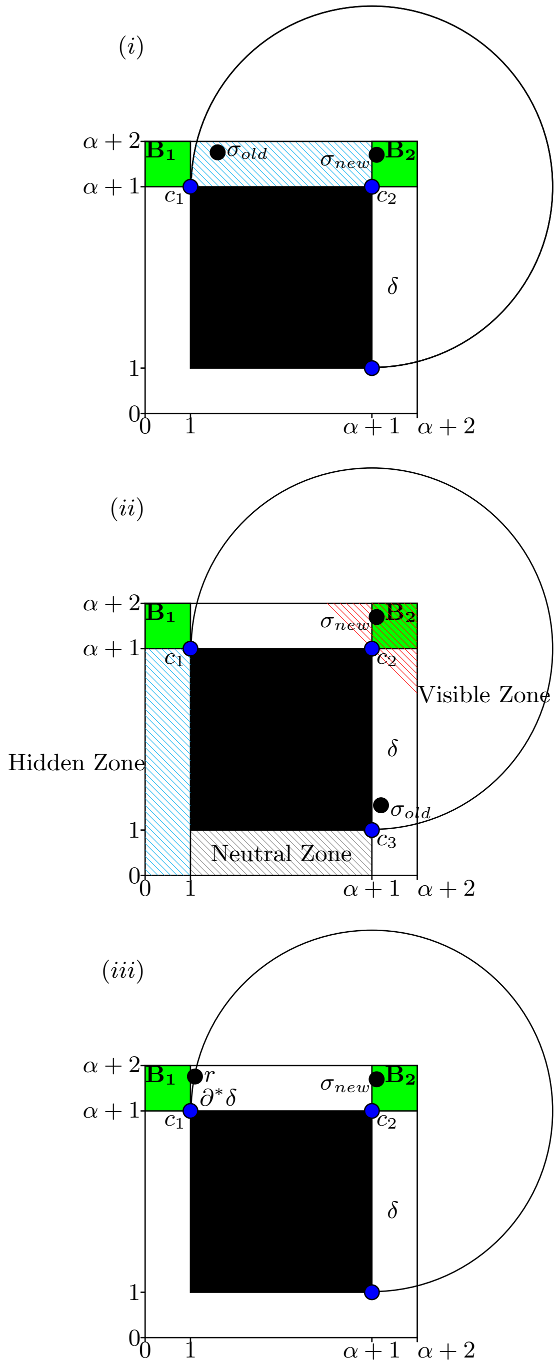

Fig. 3: Let δ denote the ℓ_2-disc around c_2 with c_1 on its boundary and let S_{cur} denote the swath of the current Bi-RRT algorithm iteration over A_α where $\alpha \geq 2$. (i) For any point $\sigma_{old} \in \mathcal{C}_{\text{free}}$ within the intersection of δ and ${}^{quad}c_2$, the top-left quadrant of c_2, σ_{old} is closer than c_1 to any given point σ_{new} in B_2; (ii) If S_{cur} intersects δ in ${c_2}_{quad}$, the bottom-right quadrant of c_2, and satisfies that $S_{cur} \cap \text{Hidden Zone} \neq \phi \implies S_{cur} \cap \text{Visible Zone} \neq \phi$, then extending T_{cur} towards a new sample σ_{new} within B_2 - the right green(solid) square, adds the sample σ_{new} to T_{cur} as a stopping configuration; (iii) For any point r on the arc $\partial^*\delta$, r is closer to B_1 - the left green(solid) square - than to c_1.

${}^{quad}c_2$ and ${c_2}_{quad}$ denote the top-left and bottom-right quadrants of c_2, resp.. Define Hidden Zone *as the rectangle* $[0,1] \times [0, \alpha+1]$, Neutral Zone *as the rectangle* $[1, a+1] \times [0,1]$ *and* Visible Zone *as the triangle* $(\alpha, \alpha+2)(\alpha+2, \alpha+2)(\alpha+2, \alpha)$. *Then,*

(i) For any point $\sigma_{old} \in \mathcal{C}_{\text{free}}$ within the intersection of δ and ${}^{quad}c_2$, σ_{old} is closer than c_1 to any given point σ_{new} in B_2.
(ii) If $S \subset \mathcal{C}_{\text{free}}$ satisfies that S intersects $\delta \cap {c_2}_{quad}$ and

$$S \cap \textit{Hidden Zone} \neq \phi \implies S \cap \textit{Visible Zone} \neq \phi, \tag{1}$$

then extending S towards a new sample σ_{new} within the B_2 region (the green(solid) square in Figure 3), as defined by Bi-RRT$_{\ell_p}(\mathcal{P}_\alpha)$, adds σ_{new} to S.
(iii) Let $\partial^*\delta$ denote the boundary of δ that lies within $\mathcal{C}_{\text{free}}$ and ${}^{quad}c_2$. For any point r on $\partial^*\delta \backslash c_1$, r is closer to B_1 than to c_1.

Proof. **[of Lemma 5]** Assume that $\alpha \geq 2$ and $1 \leq p \leq \infty$ are as mentioned in the Lemma. Assume w.l.o.g. by symmetry of the workspace, that c_1 and c_2 are the top-left and top-right, resp., corners of the square obstacle, as depicted in Figure 3. Let δ denote the open ℓ_p-disc with radius α around c_2 and let $\partial^*\delta$ be as in the Lemma (notice that by definition $c_1 \in \partial^*\delta$).

(i) **Observation 6** *For any $1 < p < \infty$ and any two points $a, b \in \mathbb{R}^2$ satisfying $a_x < b_x$ and $a_y < b_y$ the ℓ_p-bisector of (a,b), is strongly monotonically decreasing in x as a function of y. For $p = \infty$ the same holds with strongly monotonically relaxed by weakly monotonically. For $p = 1$ we further relax - for any two points $a, b \in \mathbb{R}^2$ satisfying $b_x - a_x > \ell_p{-}dist(a,b)/2 > 0$ the ℓ_p-bisector of (a,b) is weakly monotonically decreasing in x as a function of y.*
Note that generally for $p = 1$ and $p = \infty$ the bisector may be of full dimension rather than a simple curve.
Using Observation 6 for any $1 \leq p \leq \infty$ and any point $\sigma_{old} \in \delta$, the ℓ_p-bisector of (c_1, σ_{old}), is monotonically decreasing in x as a function of y and by definition $\ell_p{-}dist(c_2, c_1) > \ell_p{-}dist(c_2, \sigma_{old})$. Hence the bisector passes below or completely to the left of c_2, proving that any point σ_{new} in B_2, the top-right quadrant of c_2, is closer to σ_{old} than to c_1.
(ii) Assume on the contrary that there exists a swath $S \subset \mathcal{C}_{\text{free}}$ with $\sigma_{old} \in S \cap \delta \cap {c_2}_{quad}$ such that extending S towards σ_{new} does not add $\sigma_{new} \in B_2$ to S. Let σ_{near} be the ℓ_p-nearest point of S_{cur} to σ_{new}. By the geometry of the workspace, σ_{near} is invisible to σ_{new} and therefore lies either in Hidden Zone or in Neutral Zone. Using (mirror image of) Part (i) any point $q \in$ Neutral Zone satisfies

$$\ell_p{-}dist(q, \sigma_{new}) > \ell_p{-}dist(c_3, \sigma_{new}) \geq \ell_p{-}dist(\sigma_{old}, \sigma_{new})$$

implying that $\sigma_{near} \notin S \cap$ Neutral Zone. Hence, there exists a point $q \in S \cap$ Hidden Zone that is not visible from σ_{new}. By inference (1)

$S \cap$ Visible Zone $\neq \phi$. It remains as an easy exercise for the reader to prove that for any $\alpha \geq 2$, any point $\sigma_{new} \in interior(B_2)$ and any point $s \in$ Visible Zone

$$\ell_p - dist(q, \sigma_{new}) \geq \ell_p - dist(c_1, \sigma_{new}) > \ell_p - dist(s, \sigma_{new}),$$

a contradiction to the definition of σ_{near} as closest point.

(iii) Let r denote a point on $\partial^* \delta \backslash c_1$. Then for any $1 \leq p < \infty$ the point $u := (1, q_y)$, is the closest point to r in B_1. In particular it is closer than c_1. For $p = \infty$, r itself belongs to B_1, and is closer to B_1 than to c_1.

Lemma 7. *Let $\alpha \geq 2$ and $1 \leq p \leq \infty$ and let (c_1, c_2) denote two adjacent corners of the inner-square in $\mathbb{A}_\alpha$. Let $q_1, q_2 \in \mathcal{C}_{\text{free}}$ denote an initial and goal configurations between A_1 and B_1, and A_2 and B_2, resp., as depicted in Figure 1. Given a sequence of tree labels $\Theta_m = (\theta_1, \theta_2, \ldots, \theta_m) \in \{\hat{T}_1, \hat{T}_2\}^m$ together with a sequence of samples $\Sigma_m = (\sigma_1, \sigma_2, \ldots, \sigma_m)$, we let $W(\Theta_m, \Sigma_m)$ denote the word $\overline{w_1 w_2 \ldots w_m}$, where $w_t = (\theta_t, \sigma_t)$.*

(i) If $\overline{w_1 w_2 \ldots w_{t_i}}$ is a minimal prefix of the word W that moves s_{init} to s_i, then σ_{t_i} is added to S_1,

(ii) If $\overline{w_1 w_2 \ldots w_{t_{iii}}}$ is a minimal prefix of the word W that moves s_{init} to s_i, and then to s_{iii}, then $\sigma_{t_{iii}}$ is added to S_1, and

(iii) If $\overline{w_1 w_2 \ldots w_{t_{accept1}}}$ is a minimal prefix of the word W that moves s_{init} to s_i then to s_{iii}, and then finally to $s_{accept1}$, then $\sigma_{t_{accept1}}$ is added to both S_1 and S_2.

Proof. [**Lemma 7**] Let $q_1 \in \mathcal{C}_{\text{free}}$ be the initial configuration between A_1 and B_1 as depicted in Figure 1. Let $\Theta_m = (\theta_1, \theta_2, \ldots, \theta_m) \in \{\hat{T}_1, \hat{T}_2\}^m$ be a sequence of tree labels and $\Sigma_m = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ be a sequence of sampling.

(i) Let $W^i = \overline{w_1 w_2 \ldots w_{t_i}}$ be a minimal prefix of the word $W(\Theta_m, \Sigma_m)$, that moves s_{init} to s_i. Let S_{cur} be the induced swath by reading W^i into $\mathbb{A}_\alpha$. Since we moved by W^i to the left neighbor of s_{init} then $\theta_i = \hat{T}_1$ and $S_{\text{cur}} = S_1$. As $\mathbb{A}_\alpha$ is an ASD, and as $s_i \in States(\mathbb{A}_\alpha)$ is non-rejecting then W^i realizes $D_\sigma[s_i]$. Let δ denote the ℓ_p disc of radius α around c_1. Clearly $q_1 \in \delta$ as for any vector $v \in \mathbb{R}^2$ and any $1 \leq p \leq \infty$ $\ell_1 - norm(v) \leq \ell_p - norm(v)$ and since q_1 was picked between A_1 and B_1. Using (the mirror image of) Lemma 5 Part (ii), it is enough to prove that if S_{cur} intersects the *Hidden Zone* $(1, \alpha+2) \times (0, 1)$, then it also intersects the *Visible Zone* triangle $(0, \alpha)(0, \alpha+2)(2, \alpha+2)$; proving that the nearest point σ_{near} of S_{cur} is visible from σ_{new}. By construction of the workspace removing A_1 and the Visible Zone breaks $\mathcal{C}_{\text{free}}$ into two disconnected zones, with $q_1 \in S_{\text{cur}}$ ($q_1 \in S_1$ by definition) in the left zone and $S_{\text{cur}} \cap$ Hidden Zone $\neq \phi$ in the right zone. Since W^i realizes s_i then none of the first t_i'th samples in Σ_m intersects $R(D_\sigma[s_i])$ and using Lemma 3 $S_{\text{cur}} \cap R(D_\sigma[s_i]) = \phi$. Hence $S_{\text{cur}} \cap$ Visible Zone $\neq \phi$ and it follows that the first sample $\sigma_{t_i} \in \Sigma_m$ that intersects B_1 is added to S_1.

(ii) Let $W^{iii} = \overline{w_1 w_2 \ldots w_{t_{iii}}}$ be a minimal prefix of the word $W(\Theta_m, \Sigma_m)$, that moves along $s_{init} \rightarrow s_i \rightarrow s_{iii}$. Let S_{cur} be the induced swath by reading W^{iii} into $\mathbb{A}_\alpha$. Since we moved by W^{iii} to the left neighbor of s_i then $\theta_{iii} = \hat{T}_1$ and $S_{\text{cur}} = S_1$. As $\mathbb{A}_\alpha$ is an ASD, and as s_{iii} is non-rejecting then W^{iii} realizes $D_\sigma[s_{iii}]$. Hence, $\sigma_{t_{iii}} \in F_1(D_\sigma[s_{iii}])$. Using Part (i), $S_1 \cap \Delta^+(s_i) \neq \phi$. Let L_{top} (resp. L_{bottom}) denote the ℓ_p-bisector of c_1 and the top-(bottom-, resp.) left corner of $F_1(D_\sigma[s_i])$. Using Lemma 4, any point in $F_1(D_\sigma[s_{iii}])$ is closer to any point in $F_1(D_\sigma[s_i])$ than to c_1. Also, since $F_1(D_\sigma[s_{iii}])$ is on the left half of $D_\sigma[s_{iii}]$, any point in $F_1(D_\sigma[s_{iii}])$ is closer to c_1 than to c_2. Hence, the nearest neighbor in S_{cur} to $\sigma_{t_{iii}} \in F_1(D_\sigma[s_{iii}])$ is closer than both c_1 and c_2 and therefore visible from $\sigma_{t_{iii}}$. This proves this part of the Lemma.

(iii) Let $q_2 \in \mathcal{C}_{\text{free}}$ be the goal configuration between A_2 and B_2 as depicted in Figure 1. Let $W^{accept1} = \overline{w_1 w_2 \ldots w_{t_{accept1}}}$ be a minimal prefix of the word $W(\Theta_m, \Sigma_m)$, that moves along $s_{init} \rightarrow s_i \rightarrow s_{iii} \rightarrow s_{accept1}$. Let S_{cur} be the induced swath by reading $W^{accept1}$ into $\mathbb{A}_\alpha$. Assume we moved by $W^{accept1}$ using the left out-edge of s_{iii}. Then $\theta_{accept1} = \hat{T}_1$ and $S_{\text{cur}} = S_1$. As $\mathbb{A}_\alpha$ is an ASD, and as $s_{accept1}$ is non-rejecting then $W^{accept1}$ realizes $D_\sigma[s_{accept1}]$. Hence, $\sigma_{t_{accept1}} \in F_1(D_\sigma[s_{accept1}])$. Using Part (ii), $S_1 \cap \Delta^+(s_{iii}) \neq \phi$. Note that $\Delta^+(s_{accept1})$ was constructed such that it is contained in the disc δ with radius α around c_2. Using Lemma 5 Part (ii), any point in $\Delta^+(s_{accept1})$ is closer to any point in $F_1(D_\sigma[s_{iii}])$ than to c_1 and therefore is added to S_1. Using the same Lemma, since q_2 is between A_2 and B_2 then $\sigma_{t_{accept1}}$ is connected also to S_2. Hence, the last sample connected both trees. This proves the last part of the Lemma.

The other cases of moving using the right out-edges follows on the same line of reasoning.