Localization with Two Distance Measurements: Algebraic Analysis

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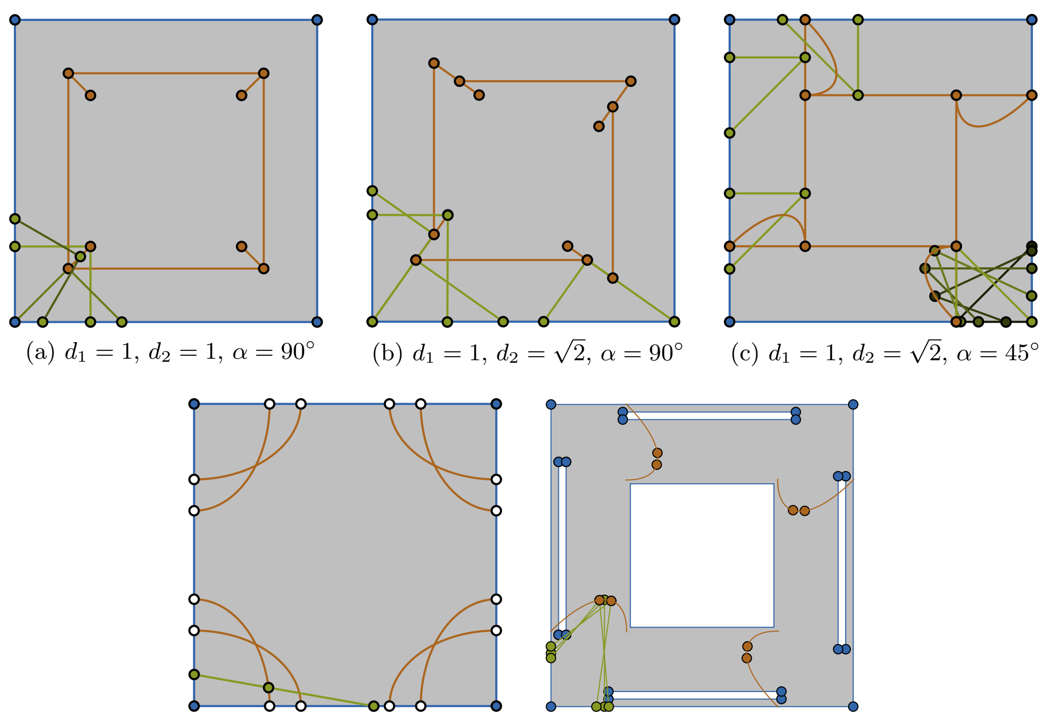
Abstract

A robot is placed inside a known polygonal workspace but in an unknown position and orientation. The robot is equipped with a distance sensor, namely, a device that can measure the distance from the sensor to the nearest object in a given direction. For simplicity, let's assume that we only concern ourselves with the position (x, y) of the sensor (rather than of the whole robot) and the orientation θ of the ray that measures the distance, namely, the angle between the ray and the positive x-axis. Our goal is to determine where in the workspace our sensor could be, after carrying out two distance measurements.

1 Introduction



(a) $d_1 = 1$, $d_2 = 1$, $\alpha = 90^\circ$ (b) $d_1 = 1$, $d_2 = \sqrt{2}$, $\alpha = 90^\circ$ (c) $d_1 = 1$, $d_2 = \sqrt{2}$, $\alpha = 45^\circ$

(d) $d_1 = 1$, $d_2 = \sqrt{2}$, $\alpha = 180^\circ$ (e) $d_1 = 1$, $d_2 = \sqrt{2}$, $\alpha = 45^\circ$

Figure 1: Various examples. The free space is filled with a light-gray color. The boundary of the free space is drawn with blue segments. Orange curves contains all the possible locations of the sensor. A pair of green segments with a common endpoint shows a witness.

If only a single measurement is carried out at an unknown direction, the possible locations of the robot comprise two-dimensional regions. Here, we concern ourselves with a variant of the problem where a query consists of three real numbers d_1, $\alpha \neq 0$, d_2 describing the following sequence of events: The sensor at its original state obtains the distance reading d_1, then the sensor is rotated (without translating) by α radians counterclockwise, and then it obtains a second distance reading d_2. The possible locations of the robot in this case comprise of one-dimensional curves in the general case; see Figure 1. If the query is augmented by a second rotation followed by a third measurement, the possible locations of the robot consist of one or more isolated points (in the general case).

A more complicated problem allows for a translation of the robot before the second measurement is taken. In this variant a query consists of four real numbers d_1, α, t, d_2, where t denotes a translation vector in the plane. If $t \neq 0$, the two measurements can be taken simultaneously. This is possible in practice, if two distinct sensors are at our disposal. We made experiments with a real robot equipped with two sensors.

2 Algebraic Analysis

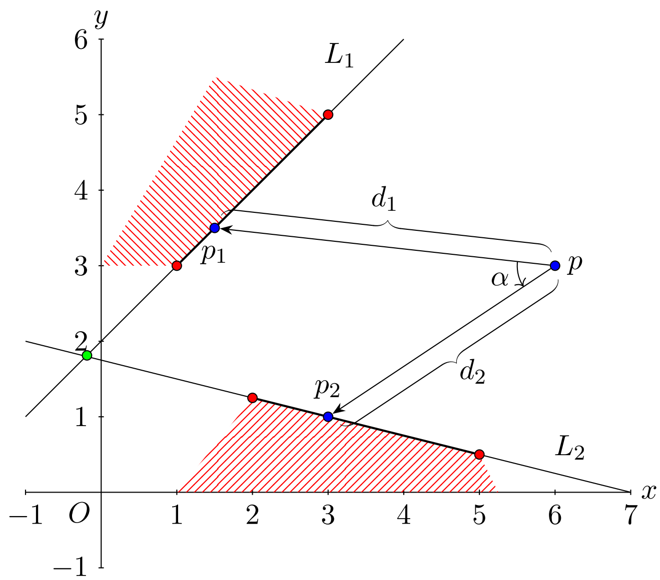


Figure 2: Local view of the problem.

We start with an algebraic analysis of the problem. Here, we concentrate at a local view of the problem, where we only consider two walls (edges) of obstacles and ignore everything else; see Figure 2. Let $L_1 : a_1x + b_1y + c_1 = 0$ and $L_2 : a_2x + b_2y + c_2 = 0$ denote the two underlying lines of the two edges of obstacles hit by the two measuring rays, respectively. Let $p = (x, y)$ denote a point in the workspace our sensor could be located at. Let $p_1 = (x_1, y_1)$ denote the point on L_1 hit by the first measuring ray, and similarly, let $p_2 = (x_2, y_2)$ denote the point on L_2 hit by the second measuring ray. Employing the law of cosine (Equation 5), the following equations must be satisfied:

$$a_1 \cdot x_1 + b_1 \cdot y_1 + c1 = 0 \quad (1)$$
$$a_2 \cdot x_2 + b_2 \cdot y_2 + c_2 = 0 \quad (2)$$
$$|p - p_1| = d_1 \quad (3)$$
$$|p - p_2| = d_2 \quad (4)$$
$$d_1^2 + d_2^2 + 2 \cdot \cos(\alpha) \cdot d_1 \cdot d_2 = |p_1 - p_2|^2 \quad (5)$$

In the degenerate case, where L_1 and L_2 are parallel (or $L_1 == L_2$) the loci of all points that satisfy the above equations form a line. This line must be trimmed to a segment according to the actual edge lengths and interiors of the obstacles. In all other cases the loci of points form two ellipses that correspond to clockwise and counter clockwise rotations. The ellipse that corresponds to the clockwise location is discarded and the other must be trimmed to obtain that actual possible locations.

Solving the above non-linear equation system is messy. We can add some constraints to the above system without compromising. First, we translate the scene, by setting $c_1 = 0$, $c_2 = 0$, coercing the intersection point of L_1 and L_2 to be at the origin. Then, we rotate the scene, setting $a_1 = 0$, coercing L_1 to lie on the x-axis. Once we find the desired elliptic arc in our transformed space, we can apply an inverse rotation followed by an inverse translation to obtain the elliptic arc in the original space.

First, we handle the simple case, where $b_2 = 0$. Recall that L_1 lies on the x-axis. This additional constraint implies that L_2 lies on the y-axis. (L_1 and L_2 are orthogonal.) Manipulating the system equation above (see Section 3.1, we obtain the single equation below; see Equation 16

$$d_1^4 \cdot x^4 + (4 \cdot k^2 - 2 \cdot d_1^2 \cdot d_2^2) \cdot x^2 \cdot y^2 - 2 \cdot d_1^2 \cdot k^2 \cdot x^2 + d_2^4 \cdot y^4 - 2 \cdot d_2^2 \cdot k^2 \cdot y^2 + k^4 = 0, \tag{6}$$

where $k = \cos(\alpha) \cdot d_1 \cdot d_2$. Let P_1 denote the bivariate polynomial on the left hand side of Equation 6. The zero set of P_1 represents the points that sautisfy the equation system above. Employing Matlab to factorize the polynomial P_1, we get: $P_1 : (A_1 \cdot x^2 + B_1 \cdot y^2 + C_1 \cdot x \cdot y + D_1) \cdot (A_2 \cdot x^2 + B_2 \cdot y^2 + C_2 \cdot x \cdot y + D_2)$, where

$A_1 = d_1^2$	$A_2 = d_1^2$
$B_1 = d_2^2$	$B_2 = d_2^2$
$C_1 = 2(d_1^2 \cdot d_2^2 - k^2)(1/2)$	$C_2 = -2(d_1^2 \cdot d_2^2 - k^2)(1/2)$
$D_1 = -k^2$	$D2 = -k^2$

The zero set of the two factors represets two ellipses, respectively. You can visualize the two ellipses and how they dynamically change as a consequence of changing the parameters d_1, $\cos\alpha$, d_2 using the GeoGebra[1] online tool; download the GeoGebra script `https://www.geogebra.org/calculator/rvqwxqcd` and upload in GeoGebra. Second, we denote $m_2 = \frac{a_2}{b_2}$, assuming the lines are not parallel. The derivation process (see Section 3.3) and the obtained polynomial (see Equation 25) in this case are much more complex. Use the GeoGebra script `https://www.geogebra.org/calculator/ferztmgh` to visualize the ellipses in this case.

3 Derivation

We repeat the system equation in the general case:

$$a_1 x_1 + b_1 y_1 + c_1 = 0 \tag{7}$$
$$a_1 x_2 + b_1 y_2 + c_1 = 0 \tag{8}$$
$$(x - x_1)^2 + (y - y_1)^2 = d_1^2 \tag{9}$$
$$(x - x_2)^2 + (y - y_2)^2 = d_2^2 \tag{10}$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d_1^2 + d_2^2 - 2cd_1d_2 = d_1^2 + d_2^2 - 2k \tag{11}$$

3.1 Constraining the Slopes of Both Lines

We set $y_1 = 0$ and $x_2 = 0$, coercing L_1 and L_2 to lie on the x- and y-axes, respectively. Equations 9, 10, and 11 reduce to:

$$(x - x_1)^2 + y^2 = d_1^2 \tag{12}$$
$$x^2 + (y - y_2)^2 = d_2^2 \tag{13}$$
$$x_1^2 + y_2^2 = d_1^2 + d_2^2 - 2k \tag{14}$$

We substitute d_1 and d_2 in Equation 14 and obtain the single equation:

$$x^2 + y^2 - xx_1 - yy_2 - k = 0 \tag{15}$$

[1] `https://www.geogebra.org`

We introduce the symbols s_1 and s_2, and get

$$x_1 = x \pm \sqrt{s_1}$$
$$y_2 = y \pm \sqrt{s_2},$$

where $s_1 = d_1^2 - y^2$ and $s_2 = d_2^2 - x^2$ based on equations 12 and 13.
We substitute x_1 and x_2 in Equation 15 and get:

$$x^2 + y^2 - x(x \pm \sqrt{s_1}) - y(y \pm \sqrt{s_2}) - k = 0$$

We expand and exchange and get:

$$\pm x\sqrt{s_1} \pm y\sqrt{s_2} = k$$

We raise to the power of two each side and get:

$$x^2 s_1 + y^2 s_2 + 2xy\sqrt{s_1 s_2} = k^2$$

We exchange and get:

$$2xy\sqrt{s_1 s_2} = k^2 - x^2 s_1 - y^2 s_2$$

We raise again and get:

$$4x^2 y^2 s_1 s_2 = (k^2 - x^2 s_1 - y^2 s_2)^2$$

We expand and get:

$$4x^2 y^2 s_1 s_2 = k^4 + x^4 s_1^2 + y^4 s_2^2 - 2k^2 x^2 s_1 - k^2 y^2 s_2 + 2x^2 y^2 s_1 s_2$$

We exchange and substitute s_1 and s_2 and get:

$$k^4 + x^4(d_1^2 - y^2)^2 + y^4(d_2^2 - x^2)^2 - 2k^2 x^2 (d_1^2 - y^2) - k^2 y^2 (d_2^2 - x^2) - 2x^2 y^2 (d_1^2 - y^2)(d_2^2 - x^2) = 0$$

We expand and regroup and get:

$$d_1^4 x^4 + y^2 x^2 (4k^2 - 2d_1^2 d_2^2) - 2d_1^2 k^2 x^2 + d_2^4 y^4 - 2d_2^2 k^2 y^2 + k^4 = 0 \tag{16}$$

3.2 Constraining the Intersection Point

We set $c_1 = c_2 = 0$, coercing the intersection point of L_1 and L_2 to coincide with the origin, and denote $m_1 = \frac{a_1}{b_1}$ and $m_2 = \frac{a_2}{b_2}$; we get $y_1 = m_1 x_1$ and $y_2 = m_2 x_2$.
We substitute y_1 and y_2 in equations 9, 10, and 11 and get:

$$(x - x_1)^2 + (y - m_1 x_1)^2 = d_1^2 \tag{17}$$
$$(x - x_2)^2 + (y - m_2 x_2)^2 = d_2^2 \tag{18}$$
$$(x_2 - x_1)^2 + (m_2 x_2 - m_1 x_1)^2 = d_1^2 + d_2^2 - 2k \tag{19}$$

We expand and regroup and get:

$$(1 + m_1^2) x_1^2 - 2(x + m_1 y) x_1 + x^2 + y^2 = d_1^2 \tag{20}$$
$$(1 + m_2^2) x_2^2 - 2(x + m_2 y) x_2 + x^2 + y^2 = d_2^2 \tag{21}$$
$$(1 + m_1^2) x_1^2 + (1 + m_2^2) x_2^2 - 2(1 + m_1 m_2) x_1 x_2 = d_1^2 + d_2^2 - 2k \tag{22}$$

We substitute d_1 and d_2 in Equation 22 and obtain the single equation:

$$(1 + m_1 m_2)x_1 x_2 - (x + m_1 y)x_1 - (x + m_2 y)x_2 + x^2 + y^2 - k = 0 \tag{23}$$

We introduce the symbols r_1, s_1, r_2, and s_2, and get

$$x_1 = r_1 \pm \sqrt{s_1}$$
$$x_2 = r_2 \pm \sqrt{s_2},$$

where

$$r_1 = ((x + m_1 y)/(1 + m_1^2))$$
$$s_1 = (((x + m_1 y)^2 - (1 + m_1^2)(x^2 + y^2 - d_1^2))/(1 + m_1^2)^2)$$
$$r_2 = ((x + m_2 y)/(1 + m_2^2))$$
$$s_2 = (((x + m_2 y)^2 - (1 + m_2^2)(x^2 + y^2 - d_2^2))/(1 + m_2^2)^2)$$

We substitute x_1 and x_2 in Equation 23 and get:

$$(1 + m_1 m_2)(r_1 + \sqrt{s_1})(r_2 + \sqrt{s_2}) - (x + m_1 y)(r_1 + \sqrt{s_1}) - (x + m_2 y)(r_2 + \sqrt{s_2}) + x^2 + y^2 - k = 0$$

We expand and regroup and get:

$$(1 + m_1 m_2)r_1 r_2 + (1 + m_1 m_2)r_1\sqrt{s_2} + (1 + m_1 m_2)r_2\sqrt{s_1} + (1 + m_1 m_2)\sqrt{s_1 s_2})$$
$$- \sqrt{s_1}(x + m_1 y) - \sqrt{s2}(x + m_2 y) - r_1(x + m_1 y) - r_2(x + m_2 y) + x^2 + y^2 - k = 0$$

We exchange and get:

$$\sqrt{s_1}(r_2\ell - x - m_1 y) + \sqrt{s_2}(r_1\ell - x - m_2 y) = r1(x + m_1 y) + r_2(x + m_2 y) - x^2 - y^2 + k - \ell r_1 r_2 - \sqrt{s_1 s_2}\ell,$$

where $\ell = 1 + m_1 m_2$.

We raise to the power of two each side and get:

$$s_1(r_2\ell - x - m_1 y)^2 + s_2(r_1\ell - x - m_2 y)^2 + 2\sqrt{s_1 s_2}(r_2\ell - x - m_1 y)(r_1\ell - x - m_2 y) =$$
$$(r_1(x + m_1 y) + r_2(x + m_2 y) - x^2 - y^2 + k - \ell(r_1 r_2))^2 + \ell^2 s_1 s_2 -$$
$$2\sqrt{s_1 s_2}\ell(r_1(x + m_1 y) + r_2(x + m_2 y) - x^2 - y^2 + k - \ell(r_1 r_2))$$

We exchange and get:

$$2\sqrt{s_1 s_2}((r_2\ell - x - m_1 y)(r_1\ell - x - m_2 y) + \ell(r_1(x + m_1 y) + r_2(x + m_2 y) - x^2 - y^2 + k - \ell r_1 r_2)) =$$
$$(r_1(x+m_1 y) + r_2(x+m_2 y) - x^2 - y^2 + k - \ell r_1 r_2)^2 + \ell^2 s_1 s_2 - s_1(r_2\ell - x - m_1 y)^2 - s_2(r_1\ell - x - m_2 y)^2$$

We raise again and get:

$$4s_1 s_2((r_2\ell - x - m_1 y)(r_1\ell - x - m_2 y) + \ell(r_1(x + m_1 y) + r_2(x + m_2 y) - x^2 - y^2 + k - \ell r_1 r_2))^2 =$$
$$((r_1(x+m_1 y) + r_2(x+m_2 y) - x^2 - y^2 + k - \ell r_1 r_2)^2 + \ell^2 s_1 s_2 - s_1(r_2\ell - x - m_1 y)^2 - s_2(r_1\ell - x - m_2 y)^2)^2 \tag{24}$$

3.3 Constraining the Slope of One Line

We set $m_1 = 0$, which implies that $\ell = 1$, in Equation 24, coercing L_1 to lie on the x-axis, and get:

$$4s_1s_2((r_2 - x - m_1y)(r_1 - x - m_2y) + (r_1(x + m_1y) + r_2(x + m_2y) - x^2 - y^2 + k - r_1r_2))^2 =$$
$$((r_1(x+m_1y)+r_2(x+m_2y)-x^2-y^2+k-r_1r_2)^2+s_1s_2-s_1(r_2-x-m_1y)^2-s_2(r_1-x-m_2y)^2)^2$$

We simplify the resutimg equation using Matlab. We obtain the bivariate polynomial P_2 (see below), the zero set of which represents the points satisfying the equation. We employ Matlab yet again to factorize the polynomial P_2. The Matlab script is available at `http://acg.cs.tau.ac.il/projects/in-house-projects/localization-with-few-distance-measurements/solution.m`.

The bivariate polynomial obtained by simplifying the system equation in the case where L_1 lies on the x-axis and and the intersection point of L_1 and L_2 coincides with the origin follows.

$$\begin{aligned}
P_2 \quad : \quad & 4 \cdot k^2 \cdot y^4 - 4 \cdot k^3 \cdot y^2 + k^4 + d_1^4 \cdot d_2^4 + 2 \cdot k^4 \cdot m_2^2 + k^4 \cdot m_2^4 + d_1^4 \cdot y^4 + d_2^4 \cdot y^4 - \\
& 4 \cdot d_1^2 \cdot k \cdot y^4 - 4 \cdot d_2^2 \cdot k \cdot y^4 - 2 \cdot d_1^2 \cdot d_2^2 \cdot k^2 + 2 \cdot d_1^2 \cdot d_2^2 \cdot y^4 - 2 \cdot d_1^2 \cdot d_2^4 \cdot y^2 - \\
& 2 \cdot d_1^4 \cdot d_2^2 \cdot y^2 + 2 \cdot d_1^2 \cdot k^2 \cdot y^2 + 2 \cdot d_2^2 \cdot k^2 \cdot y^2 + d_1^4 \cdot m_2^4 \cdot x^4 + 2 \cdot d_2^4 \cdot m_2^2 \cdot y^4 + \\
& d_2^4 \cdot m_2^4 \cdot y^4 - 4 \cdot k^3 \cdot m_2^2 \cdot y^2 + 4 \cdot k^2 \cdot m_2^2 \cdot y^4 + 4 \cdot d_1^2 \cdot d_2^2 \cdot k \cdot y^2 - \\
& 4 \cdot d_2^2 \cdot k \cdot m_2^2 \cdot y^4 - 4 \cdot d_1^4 \cdot m_2^3 \cdot x^3 \cdot y - 8 \cdot k^2 \cdot m_2^3 \cdot x \cdot y^3 - 2 \cdot d_1^2 \cdot d_2^2 \cdot k^2 \cdot m_2^2 + \\
& 4 \cdot k^3 \cdot m_2 \cdot x \cdot y - 2 \cdot d_1^4 \cdot d_2^2 \cdot m_2^2 \cdot x^2 - 2 \cdot d_1^2 \cdot d_2^2 \cdot m_2^2 \cdot y^4 - 2 \cdot d_1^2 \cdot d_2^4 \cdot m_2^2 \cdot y^2 + \\
& 2 \cdot d_1^2 \cdot k^2 \cdot m_2^2 \cdot x^2 - 2 \cdot d_1^2 \cdot k^2 \cdot m_2^4 \cdot x^2 - 2 \cdot d_1^2 \cdot k^2 \cdot m_2^2 \cdot y^2 - 2 \cdot d_2^2 \cdot k^2 \cdot m_2^4 \cdot y^2 + \\
& 6 \cdot d_1^4 \cdot m_2^2 \cdot x^2 \cdot y^2 + 4 \cdot k^2 \cdot m_2^2 \cdot x^2 \cdot y^2 + 4 \cdot k^2 \cdot m_2^4 \cdot x^2 \cdot y^2 - 4 \cdot d_1^4 \cdot m_2 \cdot x \cdot y^3 - \\
& 8 \cdot k^2 \cdot m_2 \cdot x \cdot y^3 + 4 \cdot k^3 \cdot m_2^3 \cdot x \cdot y + 4 \cdot d_1^2 \cdot d_2^2 \cdot m_2^3 \cdot x \cdot y^3 - 12 \cdot d_1^2 \cdot k \cdot m_2^2 \cdot x^2 \cdot y^2 + \\
& 4 \cdot d_1^4 \cdot d_2^2 \cdot m_2 \cdot x \cdot y - 4 \cdot d_1^2 \cdot k^2 \cdot m_2 \cdot x \cdot y + 12 \cdot d_1^2 \cdot k \cdot m_2 \cdot x \cdot y^3 + \\
& 4 \cdot d_2^2 \cdot k \cdot m_2 \cdot x \cdot y^3 + 2 \cdot d_1^2 \cdot d_2^2 \cdot m_2^2 \cdot x^2 \cdot y^2 - 2 \cdot d_1^2 \cdot d_2^2 \cdot m_2^4 \cdot x^2 \cdot y^2 - \\
& 4 \cdot d_1^2 \cdot d_2^2 \cdot m_2 \cdot x \cdot y^3 + 4 \cdot d_1^2 \cdot k^2 \cdot m_2^3 \cdot x \cdot y + 4 \cdot d_1^2 \cdot k \cdot m_2^3 \cdot x^3 \cdot y + \\
& 4 \cdot d_2^2 \cdot k \cdot m_2^3 \cdot x \cdot y^3 + 8 \cdot d_1^2 \cdot d_2^2 \cdot k \cdot m_2^2 \cdot y^2 - 8 \cdot d_1^2 \cdot d_2^2 \cdot k \cdot m_2^3 \cdot x \cdot y - \\
& 4 \cdot d_1^2 \cdot d_2^2 \cdot k \cdot m_2 \cdot x \cdot y \\
= \quad & (A_1 \cdot x^2 + B_1 \cdot y^2 + C_1 \cdot x \cdot y + D_1) \cdot (A_2 \cdot x^2 + B_2 \cdot y^2 + C_2 \cdot x \cdot y + D_2),
\end{aligned} \tag{25}$$

where

$$A_1 = d_1^2 \cdot m_2^2 \qquad A_2 = d_1^2 \cdot m_2^2$$
$$B_1 = d_1^2 + d_2^2 - 2 \cdot k + d_2^2 \cdot m_2^2 + 2 \cdot m_2 \cdot e \qquad B_2 = d_1^2 - 2 \cdot k + d_2^2 + d_2^2 \cdot m_2^2 - 2 \cdot m_2 \cdot e$$
$$C_1 = 2 \cdot m_2 \cdot (k - d_1^2 - m_2 \cdot e) \qquad C_2 = 2 \cdot m_2 \cdot (k - d_1^2 + m_2 \cdot e)$$
$$D_1 = k^2 - d_1^2 \cdot d_2^2 - k^2 \cdot m_2^2 - 2 \cdot k \cdot m_2 \cdot e \qquad D_2 = k^2 - d_1^2 \cdot d_2^2 - k^2 \cdot m_2^2 + 2 \cdot k \cdot m_2 \cdot e$$

and $e = \sqrt{d_1^2 \cdot d_2^2 - k^2}$.